THEORY OF EQUATIONS

For

Degree, Honours and Post-graduate Students

OF

All Indian Universities and for various competitive examinations

By

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Co-ordinate Solid Geometry ; Theory of Equations , Vector Analysis ;
Determinants . A Text Book of Integral Calculus ; A Text Book
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FOREWORD

I have great pleasure in writing this foreword to the book 'Theory of Equations' by Prof. M. L Khanna, M. A. of Meerut College, Meerut. Prof. Khanna in an experienced teacher and has earned a high reputation as such by dint of his ability and merit. It is needless to say that any book that comes out from his pen will have something which every student of the subject will like and which will be of immense benefit to all. I have gone through this book and have found it of high order and I can safely say that it will not be inferior to any other book on the subject available in the market.

He has written the book with the sole desire of helping the students. His method of explaining the various articles and his solution of many important examples will go a long way to make the students understand clearly the subject. I strongly recommend the book to all those for whom it is meant.

Agra Collge, Agra
30. 7. 55

Dr. M. RAY, D. Sc.

Where is What ?

Chapter *Page*

Lists of Important Formulae (i—xi)

1. General Properties of Equations 1

2. Relations between the Roots and Coefficients and Elementary Symmetric Functions 19

3. Transformation of Equations 35

4. Symmetric functions of the roots 83

5. Solution of Cubic and Biquadratic 112

6. Character and Position of the Roots of an Equation 150

7. Sturm's Theorem 178

8. Solution of Numerical Equations 198

9. Delhi, Punjab, Agra University Papers 237

LISTS OF IMPORTANT FORMULAE

List A. CHAPTER I. 1—18

A General Properties of Equations.

1. **Factor theorem.** If α is a root of the equation $f(x)=0$, then the polynomial $f(x)$ is divisible by $(x-\alpha)$ (*i.e.* there will be no remainder). Conversely if the polynomial $f(x)$ be divisible by $(x-\alpha)$, then α is a root of the equation $f(x)=0$. **(P. 3)**

2. In an equation with real coefficients, **imaginary roots occur in conjugate pairs.** **(P. 4)**

3. Every equation of odd degree has at least one real root. **(P. 5)**

4. In an equation with rational coefficients **irrational roots occur in conjugate pairs.** **(P. 5)**

5. Every equation of nth degree has n roots and no more. **(P. 5)**

6. If $f(x)$ be divided by $(x-h)$, then the **remainder R=f (h)** and it will be zero if h be a root of the equation $f(x)=0$. **(P. 8)**

7. The **common roots** of two equations $f(x)=0$ and $\phi(x)=0$ are obtained by finding the H. C. F. of $f(x)=0$ and $\phi(x)=0$. **(P. 8)**

List B. CHAPTER II 19—35

Relations between the roots and coefficients.

1. If $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ be the roots of the equation

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0,$$

then $\Sigma\alpha_1=(-1)^1\frac{a_1}{a_0}, \quad \Sigma\alpha_1\alpha_2=(-1)^2\frac{a_2}{a_0},$

$$\Sigma\alpha_1\alpha_2\alpha_3=(-1)^3\frac{a_3}{a_0}.$$

and $\alpha_1\alpha_2\alpha_3\ldots\alpha_n=(-1)^n \frac{a_n}{a_0}$ **(P. 20)**

2. For the equation $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0$ whose roots be $\alpha, \beta, \gamma, \delta$, $\Sigma\alpha=-\frac{a_1}{a_0}$,

$$\Sigma\alpha\beta=(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=\frac{a_2}{a_0},$$

$$\Sigma\alpha\beta\gamma=\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=\frac{-a_3}{a_0} \text{ and } \alpha\beta\gamma\delta=\frac{a_4}{a_0}. \quad \textbf{(P. 21)}$$

List C. CHAPTER III. 35—82

Transformation of Equations

1. Equation whose roots are reciprocals of the roots of a given equation is obtained by putting $x=\frac{1}{y}$ in the given equation which in turn means that we take last coefficient to be 1st, last but one to be 2nd and so on when the given equation is complete. **(P. 35)**

2. Equation whose roots are the roots of a given equation with sign changed is obtained by putting $x=-y$ in the given equation which in turn means that we change the sign of 2nd, 4th, 6th...*i. e.* all even terms or by changing the sign of every alternate term beginning from the 2nd. **(P. 37)**

3. Equation whose roots are the roots of the given equation multiplied by a constant m is obtained by putting $x=\frac{y}{m}$ in the given equation which in turn means that we should multiply the successive terms beginning from the 2nd by m, m^2, m^3,.......m^n respectively when the equation is complete. **(P. 39)**

4. Equation whose roots are the roots of the given equation diminished (or increased) by h. In order to find the successive coefficients of the transformed equation, divide the given complete equation by $(x-h)$, the quotient again by $(x-h)$ and so on. The successive remainders that are left in the above procedure of division are the required coefficients. The first coefficient being the same as that of the given equation. **(P. 43)**

5. Binomial coefficients.

If $f(x)=a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{2!}a_2x^{n-2}+\ldots\ldots\ldots a_n=0$ be an equation of nth degree with binomial coefficients, then the equation whose roots are the roots of this equation diminished by h is

$$A_0y^n+nA_1y^{n-1}+\frac{n(n-1)}{2!}A_2y^{n-2}+\ldots\ldots\ldots\ldots+nA_{n-1}y+A_n=0$$

where $A_0=a_0$ and all other A's are the values of polynomials with binomial coefficients of 1st, 2nd, 3rd, 4th.......degree after x is replaced by h,

i.e. $A_3=(a_0x^3+3a_1x^2+3a_2x+a_3)$ for $x=h$.

$$=a_0h^3+3a_1h^2+3a_2h+a_3$$ **(P. 50)**

6. Important symbols.

$$H=a_0a_2-a_1^2=\begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}$$

$$G=a_0^2a_3-3a_0a_1a_2+2a_1^3=\begin{vmatrix} 0 & a_0 & a_1 \\ a_0 & a_1 & a_2 \\ 2a_1 & a_2 & a_3 \end{vmatrix}$$

$$I=a_0a_4-4a_1a_3+3a_2^2$$

$$J=a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3=\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}.$$

The above symbols are for those equations which are with binomial coefficients. **(P. 51)**

The following points may be noted in writing the values of the symbols :—

(i) Sum of all the +ive coefficients
$\qquad$ =sum of the all −ive coefficients.

(ii) Sum of the suffixes of the letter 'a' in each term is same *i. e.* $a_1^3=a_1a_1a_1$ and sum of suffixes is $1+1+1=3$ etc.

(iii) Sum of the exponents of letters 'a' in each term is same.

(iv) $G^2+4H^3=a_0^2(HI-a_0J)$. **Important.** **(P. 51)**

7. The cubic $a_0x^3+3a_1x^2+3a_2x+a_3=0$ is reduced to the form $z^3+3Hz+G=0$ where $z=a_0x+a_1$ and the roots of the z cubic are

$$\frac{a_0}{3}(2\alpha-\beta-\gamma),\ \frac{a_0}{3}(2\beta-\gamma-\alpha),\ \frac{a_0}{3}(2\gamma-\alpha-\beta)$$

where α, β, γ are the roots of the x-cubic.

Also $z^3+3Hz+G=a_0^2f(x)$. **(P. 52)**

8. The biquadratic $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ is reduced to the form $z^4+6Hz^2+4Gz+a_0^2I-3H^2$ where $z=a_0x+a_1$ and the roots of the z equation are

$$\frac{a_0}{4}(3\alpha-\beta-\gamma-\delta), \text{ etc.,}$$

where $\alpha, \beta, \gamma, \delta$ are the roots of the x biquadratic.

Also $z^4+6Hz^2+4Gz+a_0^2I-3H^2=a_0^3f(x)$. **(P. 54)**

9. Equation of squared difference.

The equation of squared difference of the cubic $x^3+qx+r=0$ is

$$y^3+6qy^2+9q^2y+(27r^2+4q^3)=0. \qquad \textbf{(P. 80)}$$

If the equation whose roots are α, β, γ be $a_0x^3+3a_1x^2+3a_2x+a_3=0$, then the equation whose roots are $a_0^2(\alpha-\beta)^2$ etc. is $z^3+18Hz^2+81H^2z+27(G^2+4H^3)=0$

where G^2+4H^3 is denoted by Δ and is called the discriminant of the cubic. **(P. 80)**

10. Nature of the roots of the cubic.

(i) $G^2+4H^3 < 0$ All roots real.
(ii) $G^2+4H^3 > 0$ Two imaginary.
(iii) $G^2+4H^3=0$ Two equal.
(iv) $G=0,\ H=0$ All equal. **(P. 80)**

List D: *CHAPTER IV*. 83—111

Symmetric Functions of the Roots.

1. Some standard symmetric functions of the roots α, β, γ of a cubic.

(i) $\underset{6}{\Sigma\alpha^2\beta}=\underset{3\cdot3}{\Sigma\alpha.\Sigma\alpha\beta}-\underset{3\cdot1}{3\alpha\beta\gamma}$

(ii) $\underset{3}{\Sigma\alpha^2}=\underset{9}{(\Sigma\alpha)^2}-\underset{2\cdot3}{2\Sigma\alpha\beta}$

(iii) $\underset{3}{\Sigma\alpha^3}=\underset{3\cdot3}{\Sigma\alpha.\Sigma\alpha^2}-\underset{6}{\Sigma\alpha^2\beta}$

(iv) $\underset{3}{\Sigma\alpha^2\beta^2}=\underset{9}{(\Sigma\alpha\beta)^2}-\underset{2\cdot3}{2\alpha\beta\gamma\Sigma\alpha}.$

(v) $\underset{6}{\Sigma\alpha^3\beta}=\underset{2\cdot3}{\Sigma\alpha^2.\Sigma\alpha\beta}-\underset{3}{\alpha\beta\gamma\Sigma\alpha}.$ **(P. 83)**

2. Some standard symmetric functions of the roots $\alpha, \beta, \gamma, \delta$ of a biquadratic.

(i) $\underset{12}{\Sigma\alpha^2\beta}=\underset{4\cdot6}{\Sigma\alpha.\Sigma\alpha\beta}-\underset{3\cdot4}{3\Sigma\alpha\beta\gamma},$

(ii) $\underset{12}{\Sigma\alpha^2\beta\gamma}=\underset{4\ 4}{\Sigma\alpha.\Sigma\alpha\beta\gamma}-\underset{4\cdot1}{4\alpha\beta\gamma\delta},$

(iii) $\Sigma\alpha^2\beta^2=(\Sigma\alpha\beta)^2-2\Sigma\alpha^2\beta\gamma-6\alpha\beta\gamma\delta,$

(iv) $\Sigma\alpha^3\beta=\Sigma\alpha^2.\Sigma\alpha\beta-\Sigma\alpha^2\beta\gamma,$

(v) $\Sigma\dfrac{\alpha}{\beta}=\Sigma\alpha.\Sigma\dfrac{1}{\alpha}-4,$

(vi) $\Sigma\alpha^2\beta^2\gamma^2=(\Sigma\alpha\beta\gamma)^2-2\alpha\beta\gamma\delta.\Sigma\alpha\beta.$ **(P. 90)**

3. Newton's Theorem on the sum of the powers of the roots of an equation.

The sum of the similar powers of the roots of an equation can be expressed rationally in terms of the coefficients.

$$a_0s_1+a_1=0,$$
$$a_0s_2+a_1s_1+2a_2=0,$$
$$a_0s_3+a_1s_2+a_2s_1+3a_3=0, \text{ etc.}$$

where $s_r=\alpha^r+\beta^r+\gamma^r+\ldots.$ **(P. 99)**

List E. CHAPTER V. 112—149

1. Solution of Cubic and Biquadratic.

If $a_0x^3+3a_1x^2+3a_2x+a_3\equiv A(x-p)^3+B(x-q)^3$, then p and q are the roots of

$$\begin{vmatrix} 1 & -x & x^2 \\ a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{vmatrix}=0$$

and it is called Hessian of the given cubic and its value can also be put in the form

$$\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2=0 \text{ for } y=1,$$

where $f(x, y)=a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3=0.$

2. Euler's solution of the biquadratic.

If $f(x)=a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ be reduced to the form $z^4+6Hz^2+4Gz+a_0^2I-3H^2=0$, then Euler assumed $z=\sqrt{p}+\sqrt{q}+\sqrt{r}$ for its root, where p, q, r are the roots of the **Euler's cubic**

$$t^3+3Ht^2+\left(3H^2-\frac{a_0^2I}{4}\right)t-\frac{G^2}{4}=0.$$

Putting $t+H=a_0^2\theta$, we obtain the equation

$$4a_0^3\theta^3-Ia_0\theta+J=0$$

which is called **the reducing cubic** of the given equation.

(P. 132)

3. Relations between the roots of Euler's cubic and the given biquadratic whose roots are $\alpha, \beta, \gamma, \delta$.

$$p=\frac{a_0^2}{16}(\beta+\gamma-\alpha-\delta)^2,$$

$$q=\frac{a_0^2}{16}(\gamma+\alpha-\beta-\delta)^2,$$

$$r=\frac{a_0^2}{16}(\alpha+\beta-\gamma-\delta)^2.$$

(P. 134)

4. Relations between the roots of reducing cubic and given biquadratic

$$12\theta_1=(\gamma-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta),$$

$$12\theta_2=(\alpha-\beta)(\gamma-\delta)-(\beta-\gamma)(\alpha-\delta),$$

$$12\theta_3=(\beta-\gamma)(\alpha-\delta)-(\gamma-\alpha)(\beta-\delta).$$

(P. 137)

5. The expression $\Delta=I^3-27J^2$ is called the Discriminant of the biquadratic and

$$a_0^6(\alpha-\beta)^2(\alpha-\gamma)^2(\alpha-\delta)^2(\beta-\gamma)^2(\beta-\delta)^2(\gamma-\delta)^2=256\Delta.$$

6. Descarte's Method :—

A biquadratic can be reduced to the form

$$z^4+6Hz^2+4Gz+a_0^2I-3H^2=0$$

and since its 2nd term is missing *i e.* $\alpha+\beta+\gamma+\delta=0$ or $\alpha+\beta=-(\gamma+\delta)$ and in Descarte's method we put $f(z)=(z^2-pz+q)(z^2+pz+q')$ and by comparing the coefficients and then eliminating q and q' we obtain

$$(p^2+4H)^3-4a_0^2I(p^2+4H)-16(-a_0^3J)$$

and putting $p^2+4H=4a_0^2\theta$, we get the reducing cubic

$$4a_0^3\theta^3-Ia_0\theta+J=0.$$

(P. 141)

7. Ferrari's Method.

If $f(x)=ax^4+4bx^3+6cx^2+4dx+e=0$, then

$$f(x)=\frac{1}{a}\{(ax^2+2bx+c+2a\theta)^2-(2Mx+N)^2\}.$$

(P. 144)

List F. CHAPTER VI. – 150—177

CHARACTER AND POSITION OF THE ROOTS OF AN EQUATION

1. Descarte's Rule of Signs.

An equation $f(x)=0$ cannot have more positive roots than the number of changes of signs from + to – or from – to + in terms of its first member. **(P. 150)**

2. Change of sign of f (x).

If $f(a)$ and $f(b)$ be of opposite signs then at least one or an odd number of roots of the equation $f(x)=0$ lie between a and b. In case they are of the same sign then either no real root or an even number of roots of $f(x)=0$ lie between a and b. **(P. 152)**

3. Every equation of even degree whose last term is negative and the coefficient of the first term positive has at least two real roots one +ive and one –ive. **(P. 154)**

4. If an equation has only one change of sign it must have one +ive root and no more. **(P. 155)**

5. If all the terms of an equation are +ive and the equation involves no odd powers of x, all its roots are complex. **(P. 155)**

6. If all the terms of an equation are +ive and all involves odd power of x, then zero is the only real root. **(P. 155)**

7. Multiple roots.

If an equation $f(x)=0$ has exactly m roots equal to α, then $f(x)$ and its first $(m-1)$ derivatives all vanish for $x=\alpha$ but the mth and all the following derivatives do not vanish.

Conversely. If $f(x)$ and its first $(m-1)$ derivatives vanish for $x=\alpha$, then $f(x)=0$ has n roots equal to α. The multiple roots of $f(x)=0$ are determined by finding the H. C. F. of $f(x)=0$ and $f'(x)=0$. **(P. 157)**

8. If $f(x)$ is Max. at $x=\alpha$, then $f'(\alpha)=0$ and $f''(\alpha)$ –ive. If $f(x)$ is Min. at $x=\alpha$,, then $f'(\alpha)=0$ and $f''(\alpha)$ +ive. **(P. 170)**

9. Rolle's Theorem.

Between two consecutive real roots of the equation $f(x)=0$ there lies at least one real root of the equation $f'(x)=0$. **(P. 173)**

10. Between two consecutive roots of the equation $f'(x)=0$ there lies at the most one real root of the equation $f(x)=0$ or there may not lie any. **(P. 174)**

List G. CHAPTER VII. (178—198)

Sturm's Theorem

1. Sturm's functions.

If $f(x)$ be a given function and $f'(x)$ be its first derivative and $f_2(x), f_3(x) \ldots . f_n(x)$ be the **remainders with sign changed** in the process of finding the H. C. F. of $f(x)$ and $f'(x)$, then the $(n+1)$ functions $f(x), f'(x), f_2(x), f_3(x) \ldots . f_n(x)$ are called Sturm's functions and the $(n-1)$ remainders with sign changed *i. e.* $f_2(x), f_3(x) \ldots f_n(x)$ are called **auxiliary functions.** **(P. 179)**

2. Sturm's Theorem—unequal roots.

If $f(x)$ is a polynomial and a and b be any two real numbers, then the number of distinct real roots of $f(x)=0$ lying between a and b is exactly equal to the difference between the number of changes of signs when x is put equal to a and the number when x is put equal to b in the $(n+1)$ Sturm's functions $f(x), f'(x), f_2(x) \ldots . f_n(x)$. **(P. 179)**

3. Sturm's Theorem—equal roots.

If $f(x)=0$ be an equation having equal roots and the Sturm's function be found as $f, f', f_2 \ldots . f_r$ the last of these being the H. C. F. of $f(x)$ and $f'(x)$, then the difference between the number of changes of signs when a and b are substituted in the functions is equal to the number of real roots of the equation $f(x)=0$ which lie between a and b, each multiple root being counted once only. **(P. 182)**

4. (a) If any of Sturm's function has its roots imaginary, or is a perfect square or is always positive for real values of x, we shall stop further calculation.

(b) If the last of the Sturm's functions be numerical we need not calculate it.

Put $f_{n-1}=0$ and find the value of x and put it in f_{n-2} ; then the sign of f_n will be opposite to the sign of f_{n-2}. **(P. 183-84)**

5. **Condition for all roots real.**

In case the equation has all its roots real then the leading coefficient of the $(n+1)$ Sturm's functions should be +ive.

List H. CHAPTER VIII. 198–236

Solution of Numerical Equations

1. An equation with integral coefficients and having unity for the coefficient of the first term, cannot have a commensurable root which is not an integer. The integer roots of such an equation is a factor of the absolute term and is obtained by **Newton's Method of Divisors.** **(P. 199)**

2. **Restricting the number of trial divisors in Newton's Method.**

All those trial divisors *i. e.* the factors of absolute term which when increased by one fail to divide $f(-1)$ and which when decreased by 1 fail to divide $f(1)$ should be rejected. **(P. 202)**

Repeated roots.

In case h be a double root, then h^2 must be a factor of the last coefficient and h that of the last but one. If h be a triple root then h^3, h^2 and h should be factors of last, last but one, and last but two coefficients respectively of the given equation. **(P. 203)**

Horner's Method.

See working rule. **(P. 209, 210, 213, 218)**

CHAPTER I

GENERAL PROPERTIES OF EQUATIONS

§ 1. Rational, Integral, Algebraic Functions of x.

Algebraic function of x :—A function $f(x)$ is said to be an algebraic function of x, if it involves only the operations of addition, subtraction, multiplication, division, involution and evolution with constant exponents. For example,

$$ax^2+bx+c,\ \frac{ax+b}{cx+d},\ (2x+3)(3x+4),\ \sqrt{(5x^2-2x+1)},$$

etc. are all algebraic functions of x. Functions of the form $\tan^{-1} x$, a^{3x}, $\log x$, $\sin x$, etc. are known as **transcendental** functions of x.

Integral function of x :—A function of x is said to be integral function of x, if the quantity x does not occur in the denominator of a fraction or it never appears with $-$ive power. For example, $\frac{3}{5}x^2+\frac{1}{4}x+\frac{7}{6}$ is an integral function of x. **It may be noted here that the fractional coefficients do not change the integral character of the function.** On the other hand a function $2x^2-\frac{5}{x}+7$ is not an integral function of x.

Rational function of x :—A function is said to be rational function of x, if it involves only the operation of addition, subtraction, multiplication, and division only *i.e.* x or any operand containing x is not subjected anywhere to the process of **root extraction.** Thus $3x^2+\sqrt{2x}+5$ is a rational function of x. It may be noted here that the irrational coefficients do not change the rational character of the function. On the other hand a function $x^{2/3}+x^{1/2}+6$ is an irrational function of x.

Thus $f(x) \equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$ is said to be a rational, integral, algebraic function of x where the coefficients a's are all independent of x and that the nature of those coefficients do not change the above character of the function. The above function is of nth degree (n being assumed to be +ive) is called a polynomial.

Sign of the polynomial.

For **small values of x** the sign of the polynomial is the same as that of the last term (*i.e* the lowest degree term) in it.

For **large values of x** its sign will be the same as that of the highest degree term in it.

Algebraic equation.

$$f(x)=0 \quad i.e. \quad a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0$$

is called an equation.

The above equation is said to be complete when it contains the terms involving all the powers of x from n to zero; otherwise it is said to be incomplete. An incomplete equation can be made complete by supplying the missing powers of x with zero coefficients, *e.g.*

$$x^4+3x^2+7x-9=0$$

is incomplete as the term of x^3 is missing. It can be expressed as

$$x^4+0x^3+3x^2+7x-9=0$$

which is now complete.

Root of an equation. A value of x which when substituted in the equation reduces it to an identity is called a root of the equation.

Solution of an equation :—The determination of all the roots of a given equation means the solution of the given equation.

Degree of an equation :—By degree of a given equation we mean the highest power of x occurring in the given equation. Equations of different degrees have got different names, *e.g.*

An equation of **1st degree** is said to be **linear.**
An equation of **2nd degree** is said to be **quadratic.**
An equation of **3rd degree** is said to be **cubic.**
An equation of **4th degree** is said to be **bi-quadratic** or **quartic.**
An equation of **5th degree** is said to be **quintic.**
An equation of **6th degree** is said to be **sextic.**
An equation of **nth degree** is said to be **quantic.**

Theory of Equations : Theory of equations is that branch of Mathematics which deals with the solution of equations.

§ 2. Various Theorems.

2·1. Factor theorem : *If α is a root of the equation $f(x)=0$, then the polynomial $f(x)$ is divisible by $(x-\alpha)$ (i.e. there will be no remainder).*

Conversely : If the polynomial $f(x)$ be divisible by $(x-\alpha)$ then α is a root of the equation $f(x)=0$.

Proof : We are given that α is a root of the equation $f(x)=0$.

$$\therefore\ f(\alpha)=0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots(1)$$

Now divide $f(x)$ by $(x-\alpha)$ and let the quotient be denoted by Q and remainder (which will not involve x) by R; then

$$f(x)\equiv(x-\alpha)\,Q+R \qquad \ldots \quad \ldots \quad \ldots(2)$$

Putting $x=\alpha$, in both sides of (2), we get $f(\alpha)=R$, but from (1), $f(\alpha)=0$. $\therefore$ $R=0$ *i.e.* there will be no remainder and as such $f(x)$ is divisible by $(x-\alpha)$.

Note :—It may be noted here that in case α is not a root of the equation *i e.* $f(\alpha)\neq 0$; then $f(\alpha)=R$ where R is not zero and hence **we conclude that the value of a polynomial when x be substituted for α is equal to the remainder which is left when f(x) is divided by (x − α).**

Converse. In this case. $f(x)$ is divisible by $(x-\alpha)$ *i.e.* $f(x)=(x-\alpha)\,Q$.

Putting $x=a$ in both sides of above, we get $f(\alpha)=0$ which clearly shows that α is a root of the equation $f(x)=0$.

2·2. Theorem : *In an equation with real coefficients, imaginary roots occur in conjugate pairs.* **(Agra 56, Punjab 54)** *i.e.* if $\alpha+i\beta$ $(\beta\neq 0)$ be a root of the equation $f(x)=0$ in which the coefficients are real, then $a-i\beta$ must also be a root of the equation $f(x)=0$.

Since $\alpha+i\beta$ is a root of the equation $f(x)=0$,

$$\therefore \quad f(\alpha+i\beta)=0 \qquad \ldots \quad \ldots \quad \ldots (1)$$

Let us divide $f(x)$ by $[(x-\alpha)^2+\beta^2]$ *i.e.* $[(x-\alpha)^2-i^2\beta^2]$ or by $(x-\alpha+i\beta)(x-\alpha-i\beta)$ and let the quotient be denoted by Q and the remainder which will be of first degree be denoted by $Rx+R'$.

$$\therefore \quad f(x)=[(x-\alpha)^2+\beta^2]\,Q+Rx+R'$$

or

$$f(x)=[(x-\alpha+i\beta)(x-\alpha-i\beta)]\,Q+Rx+R' \ldots \quad \ldots (2)$$

Putting $x=\alpha+i\beta$ in both sides of (2), we get

$$f(\alpha+i\beta)=0+R\,(\alpha+i\beta)+R'.$$

But from (1), $f(\alpha+i\beta)=0$.

$$\therefore \quad R\,(\alpha+i\beta)+R'=0$$

or

$$(R.\alpha+R')+iR.\beta=0 \qquad \ldots \quad \ldots \quad \ldots (3)$$

Equating real and imaginary parts in (3), we get

$$R\alpha+R'=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots(4)$$

$$R.\beta=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots(5)$$

Now β cannot be zero for otherwise the root will be real and hence from (5), we get $R=0$ and therefore from (4) we get $R'=0$.

Hence $f(x)=[(x-\alpha+i\beta)(x-\alpha-i\beta)]\,Q$.

From above we conclude that $f(x)$ vanishes for $x=\alpha-i\beta$ and as such $\alpha-i\beta$ is also a root of the equation $f(x)=0$.

Cor. *Every equation of odd degree has at least one real root.*

We have seen that imaginary roots occur in conjugate pairs and their product gives rise to a quadratic factor of the form $(x-\alpha)^2+\beta^2$ of the equation $f(x)=0$ which being of odd degree must have at least one linear factor and hence one real root.

2·3. Theorem. *In an equation with rational coefficients irrational roots occur in conjugate pairs i.e. if $a+\sqrt{b}$ (b being not a perfect square) be a root of the equation $f(x)=0$, then $(a-\sqrt{b})$ is also a root of that equation.*

Proceed exactly as in Theorem (2) by dividing $f(x)$ by $(x-a)^2-b$ *i.e.* by $(x-a)^2-(\sqrt{b})^2$ or by $(x-a-\sqrt{b})(x-a+\sqrt{b})$ etc. and ultimately in step (3) equate rational and irrational parts.

2·4. Theorem. *Every equation of nth degree has n roots and no more.*

Let us assume here that every equation of the type

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0\ldots\ldots(1)$$

has at least one root. Let α_1 be such a root of the equation

$f(x)=0$ and as such $f(x)$ must be exactly divisible by $x-\alpha_1$ [Theorem (2).]

$\therefore$ $f(x)=(x-\alpha_1)\psi_1(x)$, where $\psi_1(x)$ is a polynomial of $(n-1)$th degree. Again $\psi_1(x)=0$ must have a root say α_2 and as such $\psi_1(x)$ must be divisible by $(x-\alpha_2)$.

$$\therefore \quad \psi_1(x)=(x-\alpha_2)\psi_2(x)$$

or

$$f(x)=(x-\alpha_1)(x-\alpha_2)\psi_2(x),$$

where $\psi_2(x)$ is a polynomial of $(n-2)$th degree. Continuing as above we get $f(x)=(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_n)\psi_n(x)$ where $\psi_n(x)$ is a polynomial of $(n-n)$ *i.e.* zero degree *i.e.* it does not contain x which means that it is a constant. On comparing the coefficients of x^n on both sides, we find that

$$a_0=\psi_n(x).$$

Hence $f(x)=a_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$ ... ...(2)

Now if x be put equal to any one of the n quantities $\alpha_1, \alpha_2\ldots\ldots\alpha_n$, then $f(x)$ vanishes and hence we can say $f(x)=0$ has n roots. Again if x be given any value other than above n quantities, then $f(x)$ does not vanish and as such any other quantity cannot be a root of the equation. Thus we conclude that an equation of nth degree has n roots and no more.

Note :—In case some of the roots be equal, then the number of distinct factors in R.H.S. of (2) will not be n, but the equation will still be said to have n roots, some of which are equal. Thus if

$$f(x)=a_0(x-\alpha_1)^p(x-\alpha_2)^q(x-\alpha_3)^r\ldots$$

where $p+q+r+\ldots=n$, then the equation has p equal roots of one kind, q equal roots of another kind and so on.

2·5. Synthetic Division. *To find the quotient and remainder when a given polynomial is divided by a binomial.*

Let the polynomial be of nth degree, say

$$f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-2}x^2+a_{n-1}x+a_n$$

and the binomial be $(x-h)$ and the quotient be

$$Q=b_0x^{n-1}+b_1x^{n-2}+b_2x^{n-3}+\ldots+b_{n-2}x+b_{n-1}$$

and remainder be denoted by R. Hence we have the following identity:

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots a_{n-2}x^2+a_{n-1}x+a_n$$
$$\equiv(x-h)\,(b_0x^{n-1}+b_1x^{n-2}+b_2x^{n-3}+\ldots b_{n-2}x+b_{n-1})+R.$$

Comparing the coefficients of like powers of x in both sides of the above identity, we have the following relations:

$$a_0=b_0 \qquad \text{or} \qquad b_0=a_0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots(1)$$
$$a_1=b_1-b_0h \qquad \text{or} \qquad b_1=a_1+hb_0 \quad \ldots \quad \ldots \quad \ldots(2)$$
$$a_2=b_2-b_1h \qquad \text{or} \qquad b_2=a_2+hb_1 \quad \ldots \quad \ldots \quad \ldots(3)$$
$$a_{n-1}=b_{n-1}-hb_{n-2} \qquad \text{or} \qquad b_{n-1}=a_{n-1}+hb_{n-2}\ldots \quad \ldots \quad \ldots(4)$$
$$a_n=R-hb_{n-1} \qquad \text{or} \qquad R=a_n+hb_{n-1} \quad \ldots \quad \ldots \quad \ldots(5)$$

From the R. H. S. of the above relations we easily obtain all the coefficients of the quotient Q along with the remainder R. The above is exhibited below in a very simple way.

$$\begin{array}{l|llllll}
h & a_0 & a_1 & a_2 & a_3\ldots\ldots & \ldots\ldots a_{n-1} & a_n \\
 & & b_0h & b_1h & b_2h\ldots\ldots & \ldots\ldots b_{n-2}h & b_{n-1}h \\
\hline
 & & b_1 & b_2 & b_3\ldots\ldots & \ldots\ldots b_{n-1} & \multicolumn{1}{|l}{R}
\end{array}$$

In first line we have written the coefficients of the given polynomial. If we are to divide the polynomial by $(x-h)$ we write h in the corner as shown. Now multiply a_0 by h and we get a_0h *i. e.* equal to b_0h by relation (1) and write it below a_1. Adding a_1 and b_0h we get a_1+b_0h *i. e.* b_1 of 3rd row [from relation (2)]. Again we multiply b_1 by h and write it below a_2 and adding we get a_2+b_1h *i. e.* b_2 of 3rd row [from relation (3)]. We continue the process and ultimately we get b_{n-1} in the 3rd row which is multiplied by h and written below a_n and on addition we get

$a_n+b_{n-1}h$ *i. e.* equal to R of 3rd row [from relation (5)].

Note 1. In case h be a root of the equation $f(x)=0$ then R will be equal to zero. [§ **2 Theorem 1**] and the equation can be depressed by one dimension.

Note 2. In case h be not a root of $f(x)=0$ then

$$R=\text{value of } f(x) \text{ for } x=h \text{ } i.\ e.\ R=f(h).$$

Note 3. In case the equation be not complete *i. e.* some of the powers of x from x^n to x^0 be missing, then we should supply the missing terms by zero coefficients *i. e.* if the equation be $2x^3-3x+5=0$ we should write it as $2x^3+0x^2-3x+5=0$ and now we shall write the first row of the above process with their respective signs.

Note 4. If the division is to be done by $x+h$, then we write $-h$ in the left-hand corner. For the sake of example, let us find the quotient and remainder, when

$$2x^4-5x^2-32x+6 \text{ is divided by } x-3.$$

$$\begin{array}{c|ccccc}
3 & 2 & 0 & -5 & -32 & 6 \\
 & & 6 & 18 & 39 & 21 \\
\hline
 & & 6 & 13 & 7 & |\,27 \quad \text{remainder.}
\end{array}$$

Hence quotient is $2x^3+6x^2+13x+7$ and remainder is 27 which is the value of the given quartic for $x=3$.

2·6. Common roots. Let there be two equations $f(x)=0$ of degree n and $\phi(x)=0$ of degree m and they have common roots $\alpha_1, \alpha_2 .. \alpha_p$, where p is less than both n and m; then $f(x)=(x-\alpha_1)(x-\alpha_2)...(x-\alpha_p). \bar{f}(x)$ where $\bar{f}(x)$ corresponds to the product of remaining factors of $f(x)$.

Similarly $\phi(x)=(x-\alpha_1)(x-\alpha_2)...(x-\alpha_p)\,\bar{\phi}(x)$. Also $\bar{f}(x)$ and $\bar{\phi}(x)$ have no other common factor.

$$(x^2-a-b)^2=4ab \text{ or } x^4-2x^2(a+b)+(a+b)^2-4ab=0$$

or $$x^4-2x^2(a+b)+(a-b)^2=0$$

which is an equation of 4th degree as was expected, for if $\sqrt{a}+\sqrt{b}$ is a root then $\sqrt{a}-\sqrt{b}$, $-\sqrt{a}+\sqrt{b}$, $-\sqrt{a}-\sqrt{b}$ will also be the roots.

First find the biquadratic as above corresponding to the four roots $\sqrt{2}-\sqrt{5}$ etc. and then as usual the remaining first degree factor is found out to be $3x-4$ etc.

5. (a) *Form an equation of the lowest degree with real and rational coefficients, one of its roots being* $\sqrt{2}+\sqrt{3}+i$.

Let $x=\sqrt{2}+\sqrt{3}+i$ $\therefore$ $x-i=\sqrt{2}+\sqrt{3}$. Square.

$$x^2+i^2-2xi=5+2\sqrt{6}$$

or $$x^2-6=2xi+2\sqrt{6}$$ Square again,

$$x^4-12x^2+36=-4x^2+24+8xi\sqrt{6}$$

or $$x^4-8x^2+12=8xi\sqrt{6}.$$ Square again.

$$(x^4-8x^2)^2+144+24(x^4-8x^2)=-64x^2(6)$$

or $$x^8-16x^6+88x^4+192x^2+144=0.$$

(b) *Form ane equation with rational coefficients which shall have for roots the values of the expression* $\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}$ *where* $\theta_1^2=1$, $\theta_2^2=1$, $\theta_3^2=1$.

Let us suppose that $x=\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}$.

Square and put $\theta_1^2=1$ etc.

$$\therefore \quad x^2-(p+q+r)=2\theta_1\theta_2\sqrt{(pq)}+2\theta_2\theta_3\sqrt{(qr)}+2\theta_3\theta_1\sqrt{(rp)}.$$

Squaring again and putting $\theta_1^2=\theta_2^2=\theta_3^2=1$, we get

$$x^4-2x^2(p+q+r)+(p+q+r)^2=4(pq+qr+rp)$$
$$+8\theta_1\theta_2\theta_3\sqrt{(pqr)}\,[\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}]$$

$$[x^4-2x^2(p+q+r)+p^2+q^2+r^2-2pq-2qr-2rp]^2=64.pqr.x^2$$

as $\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}=x$ and $\theta_1^2=1$ etc.

***6.** *Prove that the equation*

$$\frac{A^2}{x-a}+\frac{B^2}{x-b}+\frac{C^2}{x-c}+\ldots+\frac{K^2}{x-k}=x+l,$$

where A, B, C.., a, b, c.., and l are all real numbers, has all its roots real.

Let us suppose $\alpha+i\beta$ is an imaginary root of the above equation ; in that case $\alpha-i\beta$ will also be a root. Substituting for x both these roots, we get

$$\frac{A^2}{(\alpha-a)+i\beta}+\frac{B^2}{(\alpha-b)+i\beta}+\frac{C^2}{(\alpha-c)+i\beta}+\dots+\frac{K^2}{(\alpha-k)+i\beta}=(\alpha+l)+i\beta$$

and
$$\frac{A^2}{(\alpha-a)-i\beta}+\frac{B^2}{(\alpha-b)-i\beta}+\frac{C^2}{(\alpha-c)-i\beta}+\dots\dots+\frac{K^2}{(\alpha-k)-i\beta}=(\alpha+l)-i\beta.$$

Subtracting the above two relations, we get

$$-2i\beta\left[\frac{A^2}{(\alpha-a)^2+\beta^2}+\frac{B^2}{(\alpha-b)^2+\beta^2}+\frac{C^2}{(\alpha-c)^2+\beta^2}+\dots\dots+\frac{K^2}{(\alpha-k)^2+\beta^2}\right]=2i\beta$$

or
$$-2i\beta\left[\frac{A^2}{(\alpha-a)^2+\beta^2}+\frac{B^2}{(\alpha-b)^2+\beta^2}+\frac{C^2}{(\alpha-c)^2+\beta^2}+\dots\dots+\frac{K^2}{(\alpha-k)^2+\beta^2}+1\right]=0.$$

The expression within the brackets is the sum of +ive quantities and as such cannot be zero and hence $2i\beta=0$ or $\beta=0$, and hence the given equation cannot have imaginary roots. Therefore all its roots are real

(b) *Show that the equation*

$$\frac{A^2}{x-a}+\frac{B^2}{x-b}+\frac{C^2}{x-c}+\dots+\frac{H^2}{x-h}=K \text{ has all real roots.}$$

(Pb. 54)

7. *If $\alpha_1, \alpha_2 \dots \alpha_n$ be the roots of the equation*

$$(\beta_1-x)(\beta_2-x)\dots(\beta_n-x)+A=0,$$

find the equation whose roots are $\beta_1, \beta_2 \dots \beta_n$.

Since $\alpha_1, \alpha_2 \ldots \alpha_n$ are the roots of the given equation we have

$$(\beta_1-x)(\beta_2-x)\ldots\ldots(\beta_n-x)+A\equiv(\alpha_1-x)(\alpha_2-x)\ldots(\alpha_n-x).$$

$$\therefore\quad (\beta_1-x)(\beta_2-x)\ldots(\beta_n-x)=(\alpha_1-x)(\alpha_2-x)\ldots(\alpha_n-x)-A$$

showing that $\beta_1, \beta_2 \ldots \beta_n$ are the roots of

$$(\alpha_1-x)(\alpha_2-x)\ldots(\alpha_n-x)-A=0.$$

8. *The equation* $(x-1)^3+(2x-1)^3+\ldots\ldots+(nx-1)^3=0$ *has for its root* $\frac{2}{n+1}$; *find the quadratic equation satisfied by the other two roots.*

The given equation is $x^3.\Sigma n^3-3x^2.\Sigma n^2+3x.\Sigma n-n=0$.

or $$x^3\left[\frac{n(n+1)}{2}\right]^2-3x^2\left[\frac{n(n+1)(2n+1)}{6}\right]+3x\left[\frac{n(n+1)}{2}\right]-n=0.$$ Since $\frac{2}{n+1}$ is a root of the above equation $x-\frac{2}{n+1}$ will divide it without remainder and hence by synthetic division, we have

$$\begin{array}{r|cccc} \frac{2}{n+1} & \frac{n^2(n+1)^2}{4} & -\frac{n(n+1)(2n+1)}{2} & \frac{3}{2}.n(n+1) & -n \\ & & \frac{n^2(n+1)}{2} & -n(n+1) & n \\ \hline & & -\frac{n(n+1)^2}{2} & \frac{1}{2}n(n+1) & \times \end{array}$$

Hence the required quadratic equation is

$$\frac{n^2(n+1)^2}{4}x^2-\frac{n(n+1)^2}{2}x+\tfrac{1}{2}n(n+1)=0.$$

9. *Prove that* $(1-a)(1-b)(1-c)\ldots=n$, *if* $1, a, b, c\ldots$ *be the roots of the equation* $x^n-1=0$.

We have $x^n-1\equiv(x-1)[(x-a)(x-b)(x-c)\ldots]$.

Differentiate both sides w. r. t. x, treating the R. H. S. as product of two functions.

$$nx^{n-1}=1\,[(x-a)(x-b)(x-c)\ldots]$$
$$+(x-1)\frac{d}{dx}[(x-a)(x-b)(x-c)\ldots],$$

Putting $x=1$ in both sides of the above identity we get the required result.

10. *If* $f(x)=x^5-6x^4+7x^3+x^2+x+2$, *find the value of* $f(10)$.

Divide $f(x)$ by $x-10$ by synthetic division and the remainder$=f(10)$ etc. **Ans.** 47112.

11. *Find the quotient and remainder when*

$x^4+10x^3+39x^2+76x+65$ *is divided by* $x+4$.

Ans. $Q=x^3+6x^2+15x+16$, $R=1$.

12. *Find the common roots of the cubics*

$8x^3+24x^2+24x+9=0$ *and* $8x^3+72x^2+96x+45=0$

and hence find all the roots.

The H. C. F. of the two cubics is $4x^2+6x+3$ and their other factors are found out to be $(2x+3)$ and $(2x+15)$ respectively, etc.

13. *A root of the equation* $3x^3-10x^2+7x+10=0$ *is connected with a root* α' *of the equation* $x^3-x^2-17x+65=0$ *by the relation* $\alpha\alpha'+\alpha'-\alpha+1=0$. *Using this fact, solve the two given equations completely.* **(Delhi Hon's. 1948)**

Since α is a root of the given equation,

$$\therefore\quad 3\alpha^3-10\alpha^2+7\alpha+10=0\ \ldots\ \ldots\ \ldots(1)$$

Again α' is a root of the 2nd equation.

$$\therefore\quad \alpha'^3-\alpha'^2-17\alpha'+65=0\ \ldots\ \ldots\ \ldots(2)$$

But from the given relation, $\alpha'=\dfrac{\alpha-1}{\alpha+1}$. Substituting in (2), the value of α' and simplifying, we get

$$6\alpha^3+22\alpha^2+27\alpha+10=0\ \ldots\ \ldots\ \ldots(3)$$

Now $(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_p)$ divides both $f(x)$ and $\phi(x)$ and as such is the H. C. F. of $f(x)$ and $\phi(x)$. Hence in order to find the common roots of two given equations we should find their H. C. F. The ordinary procedure of finding the H. C. F. is already known to the students ; we shall exhibit the same process in a simpler way.

Ex. Show that $x^3-2x^2-2x+1=0$ and $x^4-7x^2+1=0$ have two roots common.

Note :—The letters *a*, *b*, *c*, ...denote the successive divisions below :—

Quo.		1	0	−7	0	1	4th deg.							Quo
1	(a)	1	−2	−2	1									
			2	−5	−1	1	3rd		1	−2	−2	1	3rd	
2	(b)		2	−4	−4	2			1	−3	1			1
			−1)	−1	3	−1	2nd			1	−3	1	2nd	1
				1	−3	1		(c)		1	−3	1		
											×			

Hence the H. C. F. is x^2-3x+1 which is a quadratic showing that the given equations have two common roots given by $x^2-3x+1=0$.

In the above process we have arranged the successive coefficients of the given equation (making them complete) in the parallel lines. The quotients have been written in the extreme left and right. The process of division is continued till the remainder is of lower degree than the divisor.

Note :—For the sake of convenience in division we can multiply or divide any of the lines by any number.

Ex. 2. Solve the equations

$$f(x) \equiv 4x^4+12x^3-x^2-15x=0$$

and

$$\phi(x) \equiv 6x^4+13x^3-4x^2-15x=0$$

completely, having given that they have common roots.

$$\begin{array}{|lrrrrrl|lrrrrrl|r}
 & 4 & 12 & -1 & -15 & 0 & \text{4th} & & & & & & & & \\
 & 6 & & & & & & & & & & & & & \\
\hline
 & 24 & 72 & -6 & -90 & 0 & & & 6 & 13 & -4 & -15 & 0 & \text{4th} & \\
(a) & 24 & 52 & -16 & -60 & 0 & & (b) & 6 & 3 & -9 & 0 & & & 3 \\
\hline
10) & 20 & 10 & -30 & 0 & & & 5) & 10 & 5 & -15 & 0 & & & \\
\hline
 & 2 & 1 & -3 & 0 & \text{3rd} & & & 2 & 1 & -3 & 0 & \text{3rd} & & \\
 & & & & & & & (c) & 2 & 1 & -3 & 0 & & & 1 \\
\hline
 & & & & & & & & & & \times & & & &
\end{array}$$

The H. C. F. is $2x^3+x^2-3x$.

$$\therefore\ f(x)=(2x^3+x^2-3x)(2x+5)=x(x-1)(2x+3)(2x+5)$$

$$\phi(x)=(2x^3+x^2-3x(3x+5)=x(x-1)(2x+3)(3x+5).$$

Hence the roots of $f(x)=0$ are $0, 1, -\frac{3}{2}, -\frac{5}{2}$ and those of $\phi(x)=0$, are $0, 1, -\frac{3}{2}, -\frac{5}{3}$.

2·7. *To show that if a relation of the form $\beta=\phi(\alpha)$ exists between the roots of a given equation $f(x)=0$, the equation can be depressed by two dimensions.* **(Agra 38, Pb. 41)**

Now since α and β are the roots of $f(x)=0$, we have

$$f(\alpha)=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots(1)$$

and $\qquad f(\beta)=0 \quad \text{or} \quad f[\phi(\alpha)]=0, \quad \because \quad \beta=\phi(\alpha) \ldots \quad \ldots(2)$

Hence α is a common root of (1) and (2) and can be found by the method of H. C. F. Having found the value of α, we can find the value of β from the given relation $\beta=\phi(\alpha)$. When both the roots α and β are known, the given equation $f(x)=0$ can be depressed by two dimensions by the process of synthetic division or actual division.

Ex. 3. Find all the roots of the equation

$$3x^3-10x^2+x+6=0,$$

having given that two of its roots are connected by the relation $\alpha\beta-\alpha-2$.

From the given relation we find that $\beta=\dfrac{\alpha+2}{\alpha}$.

α is a root of the equation, $\therefore\ 3\alpha^3-10\alpha^2+\alpha+6=0\ \ldots(1)$

β is a root of the equation, $\therefore\ 3\beta^3-10\beta^2+\beta+6=0\ \ldots(2)$

Put $\beta=\dfrac{\alpha+2}{\alpha}$ in (2) and we get $3(\alpha+2)^3-10\alpha(\alpha+2)^2$ $+\alpha^2(\alpha+2)+6\alpha^3=0$ which on simplification reduces to

$$5\alpha^2+\alpha-6=0\ \ldots\ \ldots\ \ldots\ \ldots(3)$$

The value of α is obtained by equating to zero the H. C. F. of (1) and (3).

	3	−10	1	6	3rd					
	5									
3	15	−50	5	30		5	1	−6	2nd	
	(a) 15	3	−18			(c) 5	−5			5
	−53	23	30		2nd	6		−6		
	−5					(d) 6		−		6
53	265	−115	−150				×			
	(b) 265	53	−318							
	−168	−168	168							
		1	−1		1st					

Hence the H. C. F. is $\alpha-1$.

$\therefore\quad \alpha=1$ and hence $\beta=\dfrac{\alpha+2}{\alpha}=3$.

The factor corresponding to these roots is $[x-1][x-3]$ or x^2-4x+3 and the remaining is easily seen to be $3x+2$ and hence the third root is $-\frac{2}{3}$.

EXERCISE I

1. *Solve the equation* $x^4+2x^3-16x^2-22x+7=0$ *having given that one of its roots is* $2+\sqrt{3}$.

We know that irrational roots of the form $a+\sqrt{b}$ occur in pairs. Hence if $2+\sqrt{3}$ is a root, then $2-\sqrt{3}$ is also a root and the product of the factors corresponding to these roots is

$$(x-2-\sqrt{3})(x-2+\sqrt{3})=(x-2)^2-3=x^2-4x+1.$$

Dividing L. H. S. of the given equation by x^2-4x+1 we get the other quadratic factor as (x^2+6x+7) which when equated to zero gives the other roots as

$$\frac{-6\pm\sqrt{(36-28)}}{2}=-3\pm\sqrt{2}.$$

2. *The equation* $3x^3-4x^2+x+88=0$ *has one of its roots* $2+\sqrt{-7}$ *; solve it completely.* **Ans.** $2\pm\sqrt{-7}, -\frac{8}{3}$.

3. **(a)** *Show that the equation* $x^6+2x^5-10x^4-12x^3+59x^2+10x-50=0$ *is satisfied by* $x=i-3$ *and* $x=i+2$ *and find the roots.* **(Punjab 22)**

Let us suppose $i-3$ or $-3+i$, is a root, then $-3-i$ is also a root. Similarly if $i+2$ or $2+i$ is a root then $2-i$ is also a root. The product of the factors corresponding to these roots is

$$[(x+3)^2-i^2][(x-2)^2-i^2]=(x^2+6x+10)(x^2-4x+5)$$
$$=(x^4+2x^3-9x^2-10x+50)$$

and we find that it divides the L.H.S. of the given equation exactly without remainder giving the quotient as (x^2-1) which corresponds to the other roots. Hence our supposition is right and the other roots are given by $x^2-1=0$ *i. e.* $x=\pm1$.

(b) *Two roots of* $x^5-x^4+8x^2-9x-15=0$ *are* $-\sqrt{3}$ *and* $1+2i$ *; find the other roots of the equation.*

Evidently other roots will be $+\sqrt{3}$ and $1-2i$ and the product of the factors corresponding to these four known roots is $(x^2-3)(x^2-2x+5)$. By actual division, the remaining factor is $x+1$. Hence the remaining root is -1.

Ans. $\sqrt{3}$, $1-2i$ and -1 are the other roots.

4. *Form an equation with rational coefficients which shall have for a root the irrational expression* $\sqrt{a}+\sqrt{b}$, *where* a *and* b *are not perfect squares and hence solve the equation* $3x^6-4x^4-42x^3+56x^2+27x-36=0$ *completely when* $\sqrt{2}-\sqrt{5}$ *is it root.*

Let $x=\sqrt{a}+\sqrt{b}$. Square both sides.

$x^2-a-b=2\sqrt{ab}$. Square again.

The H. C. F. of (1) and (3) is found out to be $3\alpha+2$ and hence $\alpha=-\frac{2}{3}$. $\therefore$ $\alpha'=-5$. Having found one root of each of the given equations, they can be depressed by one dimension and the resulting quadratics can be found giving the other roots. **Ans.** $[-\frac{2}{3}, 2\pm i]$, $[-5, 3\pm 2i]$.

14. **(a)** *If the roots of the equation* $x^4+5x^3+11x^2-13x+6=0$ *be connected by the relation* $2\beta+3\alpha=7$, *then find all its roots.*

(Pb. 41)

$$f(\alpha)=\alpha^4+5\alpha^3+11\alpha^2-13\alpha+6=0 \quad \ldots \quad \ldots \quad \ldots(1)$$

$$f(\beta)=f\left(\frac{7-3\alpha}{2}\right)=0$$

$$\therefore \quad \left(\frac{7-3\alpha}{2}\right)^4+5\left(\frac{7-3\alpha}{2}\right)^3+11\left(\frac{7-3\alpha}{2}\right)^2-13\left(\frac{7-3\alpha}{2}\right)+6=0$$

which on simplification reduces to

$$81\alpha^4-486\alpha^3+1152\alpha^2-1242\alpha+495=0 \quad \ldots\ldots \quad \ldots(2)$$

The H. C. F. of (1) and (2) is easily found out to be $\alpha-1$ and hence $\alpha=1$ and $\beta=\frac{7-3.1}{2}=2$ and the product of factors corresponding to these roots is $(x-1)(x-2)$ *i.e.* x^2-3x+2 and dividing the L. H. S. of the given equation by x^2-3x+2, we get the other factor as x^2-2x+3 which when equated to zero gives $1\pm i\sqrt{2}$ as the other two roots.

(b) *The equation* $x^3-5x^2-4x+20=0$ *has two roots whose difference is* 3. *Find all the roots of the equation.*

$\beta-\alpha=3$ $\therefore$ $\beta=\alpha+3$ *etc.* **Ans.** 2, 5, −2.

15. *The equations*

$$x^3+px^2+qx+r=0 \text{ and } x^3+p'x^2+q'x+r'=0$$

have two common roots ; find the quadratic whose roots are these two and find also the third root of each.

Let the two common roots be α and β and the quadratic factor corresponding to these be denoted by Q. If γ and γ' be the other roots of the two cubics, then

$$x^3+px^2+qx+r\equiv 0=Q\,(x-\gamma) \quad \ldots \quad \ldots \quad \ldots(1)$$

and $$x^3+p'x^2+q'x+r'\equiv 0=Q\,(x-\gamma') \ldots \quad \ldots \quad \ldots(2)$$

The coefficient of x^2 in Q should be unity. Subtracting (1) and (2), we get $(p-p')\,x^2+(q-q')\,x+(r-r')=0=Q\,(\gamma-\gamma')$.

Now when $Q.(\gamma-\gamma')=0$, then $\gamma-\gamma'\neq 0$ for otherwise $\gamma=\gamma'$ which would mean that all the three roots are equal. $\therefore$ $Q=0$ and hence making the coefficient of x^2 unity in L. H. S. we get the required quadratic corresponding to α and β as $$x^2+\frac{q-q'}{p-p'}x+\frac{r-r'}{p-p'}=0 \quad \ldots \quad \ldots \quad \ldots(3)$$

$\therefore$ $\alpha\beta=$product of the roots of $(3)=\dfrac{r-r'}{p-p'}$, $\ldots \quad \ldots(4)$

Also $\alpha\beta\gamma=$product of the roots of $(1)=-r$.

and $\alpha\beta\gamma'=$product of the roots of $(2)=-r'$.

Hence $\gamma=-r\,\dfrac{(p-p')}{(r-r')}$ and $\gamma'=-r'\,\dfrac{(p-p')}{(r-r')}$.

CHAPTER II

RELATIONS BETWEEN THE ROOTS AND COEFFICIENTS AND ELEMENTARY SYMMETRIC FUNCTIONS

§ 1. Relation between the roots and coefficients. (Nagpur 57, Pb. 31, 34, 35, 37, 44, 51, 60)

We have already read that if α and β be the roots of a quadratic equation $ax^2+bx+c=0$, then

$$\text{sum of the roots}=\alpha+\beta=\frac{-b}{a}.$$

and
$$\text{product of the roots}=\alpha\beta=\frac{c}{a}.$$

We shall now establish the above type of relations between the roots and coefficients of an eqution of nth degree.

Let the general equation of nth degree be

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0 \quad \ldots(1)$$

and its roots be $\alpha_1, \alpha_2, \alpha_3\ldots\alpha_n$; then clearly

$$f(x)\equiv K(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n).$$

Comparing the coefficients of x^n in both sides of the above identity, we get $a_0=K$, and hence

$$\begin{aligned} f(x)&\equiv a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)\\ &\equiv a_0[x^n-x^{n-1}(\alpha_1+\alpha_2+\ldots+\alpha_n)\\ &\quad +x^{n-2}(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3+\ldots)\\ &\quad -x^{n-3}(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_3\alpha_4+\ldots)+\ldots\\ &\quad +(-1)^n(\alpha_1\alpha_2\alpha_3\ldots\alpha_n)]. \end{aligned}$$

$$\equiv a_0\,[x^n - x^{n-1}\,(\Sigma\,\alpha_1) + x^{n-2}\,(\Sigma\,\alpha_1\alpha_2) - x^{n-3}\,(\Sigma\,\alpha_1\alpha_2\alpha_3) + \ldots + \ldots + (-1)^n\,(\alpha_1\alpha_2\alpha_3\ldots\alpha_n)].$$

Comparing the coefficients of like powers of x in both sides, we get

$$-a_0\Sigma\,\alpha_1 = a_1 \quad \therefore \quad \Sigma\,\alpha_1 = \frac{-a_1}{a_0} = (-1)^1\,\frac{a_1}{a_0}$$

$$a_0\Sigma\,\alpha_1\alpha_2 = a_2 \quad \therefore \quad \Sigma\,\alpha_1\alpha_2 = \frac{a_2}{a_0} = (-1)^2\,\frac{a_2}{a_0}$$

$$-a_0\Sigma\,\alpha_1\alpha_2\alpha_3 = a_3 \quad \therefore \quad \Sigma\,\alpha_1\alpha_2\alpha_3 = \frac{-a_3}{a_0} = (-1)^3\,\frac{a_3}{a_0}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_0\,(-1)^n\,(\alpha_1\alpha_2\alpha_3\ldots\alpha_n) = a_n.$$

$$\therefore \quad \alpha_1\alpha_2\alpha_3\ldots\alpha_n = \frac{a_n}{(-1)^n a_0} = (-1)^n\,\frac{a_n}{a_0} \quad \because \quad (-1)^{2n} = 1.$$

where $\Sigma\,\alpha_1$ stands for the sum of the roots and $\Sigma\,\alpha_1\alpha_2$ stands for the sum of the products of these roots taken two by two etc.

Rule—In a complete equation (if the equation be not complete, then it should first be made complete by supplying the missing powers of x by zero coefficients). Sum of the products of the roots taken p at a time

$$= (-1)^p\,\frac{\text{coefficient of } (p+1)\text{th term}}{\text{coefficient of 1st term}}$$

or **in case p be even** *i.e.* $(-1)^p = 1$, we can write the above formula as

$$\frac{\text{coefficient of } (p+1)\text{th term}}{\text{coefficient of 1st term}}$$

and **In case p be odd** *i.e.* $(-1)^p = -1$, then we can put the above formula as

$$\frac{\text{coefficient of } (p+1)\text{th term } \textbf{with sign changed}}{\text{coefficient of 1st term}}$$

Deduction : (1) If an equation whose leading coefficient a_0 be +ive has all its roots +ive, then

$$\Sigma\, \alpha_1 = -\frac{a_1}{a_0} = +\text{ive};\ \therefore\ a_1 \text{ must be } -\text{ive};$$

$$\Sigma\, \alpha_1\alpha_2 = \frac{a_2}{a_0} = +\text{ive};\ \therefore\ a_2 \text{ must be } +\text{ive}.$$

$$\Sigma\, \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0} = +\text{ive};\ \therefore\ a_3 \text{ must be } -\text{ive and so on};$$

showing that the coefficients of the above equation should be alternately +ive and −ive.

Deduction : (2) Similarly we can prove that if an equation whose leading coefficient a_0 be +ive has all its roots −ive then all the coefficients of the above equation should be +ive.

§ **2. Particular Cases :** Relations between the roots and coefficients of a cubic and a biquadratic equation.

Cubic : $a_0x^3+a_1x^2+a_2x+a_3=0$, roots being α, β, γ.

$$\Sigma\, \alpha = \alpha+\beta+\gamma = -\frac{a_1}{a_0},$$ ✓

$$\Sigma\, \alpha\beta = \alpha\beta+\beta\gamma+\gamma\alpha = \frac{a_2}{a_0},$$ ✓

$$\Sigma\, \alpha\beta\gamma = \alpha\beta\gamma = -\frac{a_3}{a_0}.$$ ✓

Biquadratic : $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0$.

$$\Sigma\, \alpha = \alpha+\beta+\gamma+\delta = -\frac{a_1}{a_0}.$$ ✓

$$\Sigma\, \alpha\beta = \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{a_2}{a_0}$$ ✓

(6 terms)

or $$\Sigma\ \alpha\beta=(\alpha+\beta)\ (\gamma+\delta)+\alpha\beta+\gamma\delta=\frac{a_2}{a_0}$$

(Note this form).

$$\Sigma\ \alpha\beta\gamma=\alpha\beta\gamma+\alpha\beta\delta+\gamma\delta\alpha+\gamma\delta\beta=-\frac{a_3}{a_0}$$

(4 terms)

or $$\Sigma\ \alpha\beta\gamma=\alpha\beta\ (\gamma+\delta)+\gamma\delta\ (\alpha+\beta)=-\frac{a_3}{a_0}$$

(Note this form).

$$\Sigma\ \alpha\beta\gamma\delta=\alpha\beta\gamma\delta=\frac{a_4}{a_0}.$$

Note :—Before doing the examples to follow, the students should remember that if we have found a root of a given equation we can depress the equation by one dimension by the process of synthethic division (Note 1, page 8). If l be the sum of the two roots and m be their product, then the quadratic equation of which these are the roots is x^2+(sum with sign changed) x+product=0 *i.e.* $x^2-lx+m=0$.

Examples

1. (a) *Find the condition which must be satisfied by the coefficients of the equation $x^3-px^2+qx-r=0$ when two of its roots α, β are connected by the relation $\alpha+\beta=0$.* **(Pb. 44)**

Let the roots of the equation be α, β, γ.

$$\Sigma\alpha=\alpha+\beta+\gamma=p, \text{ but } \alpha+\beta=0; \quad \therefore \quad \gamma=p.$$

Now γ is a root of the given equation and as such

$$\gamma^3-p\gamma^2+q\gamma-r=0. \quad \text{Put the value of } \gamma.$$

$\therefore\ \ p^3-p.p^2+q.p-r=0$ or $pq=r$ is the required condition.

(b) *The equation $x^3-5x^2-16x+80=0$ has two of its roots equal in magnitude but opposite in sign ; solve it completely.*

Ans. ±4, −5.

2. (a) *If one root of the equation* $x^3-px^2+qx-r=0$ *be (i) n times, (ii) double, (iii) equal to the other root of the equation, show that it may be found from a quadratic.*

Let the roots be α, $n\alpha$ and β.

$$\Sigma\alpha=(n+1)\,\alpha+\beta=p; \quad \therefore \quad \beta=p-(n+1)\,\alpha \quad \ldots\ldots(1)$$

$$\Sigma\alpha\beta=n\alpha^2+\alpha\beta\,(n+1)=q \quad \ldots \quad \ldots \quad \ldots(2)$$

Putting the value of β from (1) in (2), we get the required quadratic as $n.\alpha^2+(n+1)\,\alpha\,[p-(n+1)\,\alpha]=q$.

or $$(n+1)^2\,\alpha^2-n\alpha^2-p\,(n+1)\,\alpha+q=0$$

or $$(n^2+n+1)\,\alpha^2-p\,(n+1)\,\alpha+q=0 \quad \ldots \quad \ldots(3)$$

Having found α from above we can find β from (1). Since there will be two values of α and hence two of β, we should choose that set which satisfies the third relation

$$\Sigma\alpha\beta\gamma=n\alpha^2\beta=r.$$

For (ii) and (iii) put $n=2$ and 1 respectively in (3).

(b) *The cubic* $2x^3-9x^2+12x+\lambda=0$ *has two equal roots. Find* λ *and solve the equation completely.*

Ans. $1, 1, \frac{5}{2}$ when $\lambda=-5$,
$2, 2, \frac{1}{2}$ when $\lambda=-4$.

(c) *Solve the equation* $x^3-9x^2+14x+24=0$, *two of its roots being in the ratio* $3:2$.

Let the roots be 3α, 2α and β and proceeding as above, we get the quadratic $19\alpha^2-45\alpha+14=0$, giving

$$\alpha=2 \quad \therefore \quad \beta=-1,$$

or $$\alpha=\tfrac{7}{19} \quad \therefore \quad \beta=7\tfrac{3}{19}.$$

The later set of values of α and β do not satisfy the relation $\Sigma\alpha\beta\gamma$ or $6\alpha^2\beta=-24$.

Thus the roots are $6, 4, -1$. **Ans.**

3. (a) *Find the condition that the roots of the equation*

$x^3-px^2+qx-r=0$ may be in arithmetical progression and hence solve the equation $x^3-12x^2+39x-28=0$. **(Pb. 40)**

Let the roots of the given equation be $a-d$, a, $a+d$, being in A.P.

$$\therefore\quad \Sigma\alpha=a-d+a+a+d=p \quad\text{or}\quad 3a=p \quad\text{or}\quad a=\frac{p}{3}.$$

But a is a root of the given equation and as such it will satisfy it.

$$\therefore\quad \left(\frac{p}{3}\right)^3-p.\left(\frac{p}{3}\right)^2+q.\left(\frac{p}{3}\right)-r=0.$$

or $\quad 2p^3-9pq+27r=0$ is the required condition.

Proceeding as above we find that $a=4$ is a root of the given equation which when divided by $x-4$ gives the quadratic equation $(x^2-8x+7)=0$ and hence the other roots are 1 and 7 and clearly 1, 4, 7 are in A.P.

(b) *The roots of $2x^3-15x^2+37x-30=0$ are in A.P. Find them.* **(Raj. 55)**

Ans. 2, $\frac{5}{2}$, 3.

(c) *Find the condition that the roots of the equation $x^5+px^4+qx^3+rx^2+sx+t=0$ be in A.P.*

Let the roots be $a-2d$, $a-d$, a, $a+d$, $a+2d$, Clearly sum of the roots$=5a=-p$; $\therefore$ $a=-\frac{p}{5}$.

But a is a root of the given equation which will satisfy it. Hence the required condition is

$$\left(\frac{-p}{5}\right)^5+p\left(\frac{-p}{5}\right)^4+q\left(\frac{-p}{5}\right)^3+r\left(\frac{-p}{5}\right)^2+s\left(\frac{-p}{5}\right)+t=0$$

or $$4p^5-25qp^3+125rp^2-625sp+3125t=0.$$

4. (a) *Find the condition that the roots of the equation $x^3-px^2+qx-r=0$ be in geometrical progression.*

(Agra 35, 40)

Hence solve the equation $27x^3+42x^2-28x-8=0$.

Let the roots of the given equation be ap, a, $\frac{a}{p}$, being in G. P.

$$\therefore \quad \Sigma\alpha\beta\gamma=\alpha\beta\gamma=ap.a.\frac{a}{p}=r \quad \text{or} \quad a^3=r.$$

But a is a root of the given equation and as such $a^3-pa^2+qa-r=0$ or $-pa^2+qa=0$ [$\because a^3=r$] or $pa=q$ or $p^3a^3=q^3$ or $p^3r=q^3$ or $p^3r-q^3=0$ is the required condition.

Proceeding as above we find that $a=\frac{2}{3}$ is a root of the given equation which when divided by $3x-2$ gives the quadratic equation $(9x^2+20x+4)=0$ and hence the other roots are -2 and $-\frac{2}{9}$ and clearly -2, $\frac{2}{3}$, $-\frac{2}{9}$ are in G. P.

(b) *If* α, β, γ *are the roots of the cubic equation* $ax^3+3bx^2+3cx+d=0$, *find the condition that they may be in geometrical progression.* **(Agra 48)**

Ans. $b^3d-ac^3=0$.

5. (a) *Find the condition that the roots of the equation* $x^3-px^2+qx-r=0$ *be in harmonical progression.*

(Agra 34, 37, 45; Pb. 36, 51; Delhi Hon's 52)

Show that the mean root is $3r/q$.

Hence solve the equation $6x^3-11x^2-3x+2=0$. **(Pb. 54)**

Let the roots be α, β, γ which are in H. P. and hence

$$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma} \text{ are in A. P;} \quad \therefore \quad \frac{1}{\alpha}+\frac{1}{\gamma}=\frac{2}{\beta}$$

or
$$\beta\gamma+\alpha\beta=2\gamma\alpha.$$

Adding $\gamma\alpha$ to both sides, we get

$$\alpha\beta+\beta\gamma+\gamma\alpha=3\gamma\alpha \quad \text{or} \quad q=3\gamma\alpha \quad \text{or} \quad \gamma\alpha=q/3.$$

Again $\alpha\beta\gamma=r$ or $\left(\frac{q}{3}\right)\beta=r$; $\therefore$ $\beta=\frac{3r}{q}=$mean root.

But β is a root of the given equation and as such

$$\left(\frac{3r}{q}\right)^3-p\left(\frac{3r}{q}\right)^2+q\left(\frac{3r}{q}\right)-r=0$$

or $27r^2-9pqr+2q^3=0$ is the required condition.

Proceeding as above we get $3\gamma\alpha=\Sigma\alpha\beta=-\frac{3}{6}=-\frac{1}{2}$.

$\therefore$ $\gamma\alpha=-\frac{1}{6}$. Also $\alpha\beta\gamma=-\frac{2}{6}$ or $(-\frac{1}{6})\beta=-\frac{2}{6}$.

$\therefore$ $\beta=2$ is a root of the given equation which when divided by $(x-2)$ gives the quadratic $(6x^2+x-1)=0$ giving the roots as $-\frac{1}{2}$ and $\frac{1}{3}$. Hence the roots are $-\frac{1}{2}, 2, \frac{1}{3}$ which are clearly in H. P. for their reciprocals $-2, \frac{1}{2}, 3$ are in A. P.

Note :—If for every x we write $1/x$ in the given equation and simplify, then the roots of the resulting equation will be the reciprocal of the roots of the given equation. By doing so the above questions can be convenienty solved like Q. 3. **[See Ex. 1 § 2, Chap. III]**

(b) *Solve the equation* $105x^3-142x^2+60x-8=0$, *its roots being in H. P.* **(Pb. 39)**

Ans. $\frac{2}{7}, \frac{2}{5}, \frac{2}{3}$.

6. *Find the condition that the roots of the equation* $x^3+3px^2+3qx+r=0$ *may be in (i) A. P., (ii) G. P., (iii) H. P.* **(Delhi Hon's 50)**

Ans. (i) $2p^3-3pq+r=0$, (ii) $p^3r-q^3=0$,
(iii) $2q^3-3pqr+r^2=0$.

7. *The distances of three points A, B, C on a straight line from a fixed origin O on the line are the roots of the equation* $ax^3+3bx^2+3cx+d=0$; *find the condition that one of the points A, B, C should bisect the distance between the other two.* **(Delhi Hon's 51, 54)**

If α, β, γ be the roots, then we have to find the condition that $\beta=\frac{\alpha+\gamma}{2}$ *i. e.* α, β, γ be in A. P. etc. as in Q. 3. $2b^3-3abc+a^2d=0$ is the required condition.

8. *Find the condition that the equation*

$$x^4+px^3+qx^2+rx+s=0$$

should have two roots connected by the relation $\alpha+\beta=0$ *and determine in that case the two quadratics which shall have for roots (1)* (α, β), *and (2)* (γ, δ) *and hence solve completely the equation*

$$x^4-2x^3+4x^2+6x-21=0 \qquad \textbf{(Agra 35)}$$

$\alpha+\beta=0$ given.

$\Sigma\alpha=\alpha+\beta+\gamma+\delta=-p,$

$$\therefore \quad \gamma+\delta=-p. \quad \because \quad \alpha+\beta=0 \ldots \quad \ldots(1)$$

$$\Sigma\alpha\beta=(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=q \text{ or } \alpha\beta+\gamma\delta=q \ldots \quad \ldots(2)$$

$$\Sigma\alpha\beta\gamma=\alpha\beta\,(\gamma+\delta)+\gamma\delta\,(\alpha+\beta)=-r$$

$$\therefore \quad \alpha\beta\,(-p)=-r \text{ from (1).} \quad \therefore \quad \alpha\beta=r/p \ldots \quad \ldots(3)$$

$$\Sigma\alpha\beta\gamma\delta=\alpha\beta\gamma\delta=s \quad \text{or} \quad \left(\frac{r}{p}\right)\gamma\delta=s\,;$$

$$\therefore \quad \gamma\delta=\frac{ps}{r} \ldots \quad \ldots(4)$$

Substituting the values of $\alpha\beta$ and $\gamma\delta$ in (2), we get the required condition as $\dfrac{r}{p}+\dfrac{ps}{r}=q$ or $pqr-p^2s-r^2=0$.

Again since $\alpha+\beta=0$ and $\alpha\beta=r/p$ $\therefore$ α and β are the roots of the quadratic $x^2-0x+r/p=0$ or $px^2+r=0$.

Also $\gamma+\delta=-p$ and $\gamma\delta=\dfrac{ps}{r}$; $\therefore$ γ and δ are the roots of the equation $x^2-(-p)\,x+\dfrac{ps}{r}=0$ or $rx^2+prx+ps=0$.

Proceeding as above the required roots are

$$\sqrt{3},\ -\sqrt{3},\ 1\pm\sqrt{(-6)}.$$

9. *(a) Find the condition that sum of the two roots of the equation* $x^4+px^3+qx^2+rx+s=0$ *be equal to the sum of the other two roots.* **(Nagpur 57)**

Hence or otherwise, solve the equation

$$x^4-8x^3+21x^2-20x+5=0. \qquad \textbf{(Pb. 1916)}$$

$$\alpha+\beta=\gamma+\delta \quad \text{(given)} \ldots \quad \ldots \quad \ldots(1)$$

$$\Sigma\alpha=\alpha+\beta+\gamma+\delta=-p.$$

$$\therefore \quad \alpha+\beta=\gamma+\delta=-p/2 \ldots \quad \ldots \quad \ldots(2)$$

$$\Sigma\alpha\beta=(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=q$$

or $$(-p/2)(-p/2)+\alpha\beta+\gamma\delta=q$$

or $$\alpha\beta+\gamma\delta=q-p^2/4=\frac{4q-p^2}{4} \ldots \quad \ldots \quad \ldots(3)$$

$$\Sigma\alpha\beta\gamma=\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-r$$

or $$-\frac{p}{2}[\alpha\beta+\gamma\delta]=-r \quad \text{from (2)}$$

or $$-\frac{p}{2}\left[\frac{4q-p^2}{4}\right]=-r \quad \text{from (3)}$$

or $p^3-4pq+8r=0$ is the required condition.

In the given question $\alpha+\beta=\gamma+\delta=-p/2=4$ and let $\alpha\beta=l$ and $\gamma\delta=m$ and hence

$$x^4-8x^3+21x^2-20x+5\equiv(x^2-4x+l)(x^2-4x+m).$$

Comparing the coefficients of like powers of x, we get

$$-4(l+m)=-20 \; ; \quad \therefore \quad l+m=5 \quad \text{and} \quad lm=5.$$

$\therefore$ l and m are the roots of $t^2-5t+5=0$.

$$\therefore \quad t=\frac{5\pm\sqrt{(25-20)}}{2}=\frac{5\pm\sqrt{5}}{2}.$$

Hence the two quadratics are $x^2-4x+\left(\frac{5+\sqrt{5}}{2}\right)=0$

and $$x^2-4x+\left(\frac{5-\sqrt{5}}{2}\right)=0$$

which on solving give the required roots as

$$\frac{3\pm\sqrt{5}}{2}, \; \frac{5\pm\sqrt{5}}{2}.$$

Note. $\sqrt{(6\pm2\sqrt{5})}=(1\pm\sqrt{5})$

(b) *The sum of two roots of the equation*

$$x^4-8x^3+19x^2+4\lambda x+2=0$$

is equal to the sum of the other two. Find λ *and solve the equation.*

(Delhi Hon's 1949)

$\lambda=-3,\ 2\pm\sqrt{2},\ 2\pm\sqrt{3}.$ **Ans.**

(c) *Find the condition that the roots of the same equation be connected by the relation*

$$(\alpha+\beta)\,\gamma\delta=(\gamma+\delta)\alpha\beta$$

or $$\frac{\alpha+\beta}{\alpha\beta}=\frac{\gamma+\delta}{\gamma\delta} \quad \text{or} \quad \frac{1}{\alpha}+\frac{1}{\beta}=\frac{1}{\gamma}+\frac{1}{\delta}.$$

Put $x=\dfrac{1}{x}$ in the given equation and proceed as in part (a) for the transformed equation as its roots will be reciprocal of the roots of the given equation.

10. (a) *Find the condition that the roots* $\alpha, \beta, \gamma, \delta$ *of equation* $x^4+px^3+qx^2+rx+s=0$ *should be connected by the relation* $\alpha\beta=\gamma\delta$. **(Agra 1944, Pb. 1934, 52)**

Hence or otherwise solve the equation,

$$x^4+20x^3-210x^2-540x+729=0$$

$\alpha\beta=\gamma\delta$ given.

But $\alpha\beta\gamma\delta=s.$ $\therefore$ $\alpha\beta=\gamma\delta=\pm\sqrt{s}$.. ... ...(I)

$\Sigma\,\alpha=\alpha+\beta+\gamma+\delta=-p$ ... ...(2)

$\Sigma\,\alpha\beta=(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=q.$ ... ...(3)

$\Sigma\,\alpha\beta\gamma=\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-r;$

or $\pm\sqrt{s}(\alpha+\beta+\gamma+\delta)=-r$ from (1)

or $\pm\sqrt{s}(-p)=-r$ from (2).

$\therefore$ $p^2s=r^2$ is the required condition.

In the given equation, $\alpha\beta\gamma\delta=729.$ $\therefore$ $\alpha\beta=\gamma\delta=+27$ or -27. If $\alpha+\beta=l$ and $\gamma+\delta=m$, then

$$(x^4+20x^3-210x^2-540x+729\equiv(x^2-lx-27)(x^2-mx-27).$$

Comparing the coefficients of like powers of x, we get

$$-(l+m)=20 \quad \text{or} \quad l+m=-20$$

and $\quad lm-27-27=-210 \quad \text{or} \quad lm=-156.$

Also comparing x, we find $27(l+m)=-540$ or $l+m=-20$ which we have already found. Here if we take $\alpha\beta=\gamma\delta=27$, then the last relation will not give $l+m=-20$ as already established.

$\therefore$ l and m are the roots of $t^2+20t-156=0$

$$(t+26)(t-6)=0; \quad \therefore \quad t=-26 \text{ and } 6.$$

Hence the two quadratics giving the four roots are $x^2+26x-27=0$ and $x^2-6x-27=0$.

$$\therefore \quad x=1,\ -3,\ 9,\ -27.$$ **Ans.**

(b) *Find the relation between the coefficients of the equation* $x^4+px^3+qx^2+rx+s=0$ *under the following conditions :—(1) Its roots are in geometrical progression. (2) Ratio of two of the roots is equal to the ratio of the other two.*

(i) Let the roots be a, ar, ar^2, ar^3 ; then clearly $(a)(ar^3)=(ar)(ar^2)$ *i.e.*, the product of two roots is equal to the product of the other two and hence the condition is the same as that of part (*a*).

(ii) $\frac{\alpha}{\delta}=\frac{\gamma}{\beta}$ *i, e.* $\alpha\beta=\gamma\delta$ and hence etc. as in (a).

(c) *Hence solve the equation* $x^4+15x^3+70x^2+120x+64=0$.

(Agra 1942)

Proceed as in part (a). $-1,\ -2,\ -4,\ -8.$

(d) *Solve the equation* $x^4+x^3-16x^2-4x+48=0$ *having given that product of two of the roots is 6.*

If $\alpha\beta=6$, then since $\alpha\beta\gamma\delta=48$, $\therefore$ $\gamma\delta=8$ and now proceed as in part (a) by taking $\alpha+\beta=l$ and $\gamma+\delta=m$ so that

$$x^4+x^3-16x^2-4x+48=(x^2-lx+6)\,(x^2-mx+8)$$

and comparing the coefficients $l=5$, $m=-6$.

Ans. 3, 2, -2, -4.

(e) *Find the condition that the biquadratic of part (b) may have its roots connected by the relation* $\alpha\beta+1=0$.

$$\alpha\beta=-1,\ \alpha\beta\gamma\delta=s\ ;\quad \therefore\quad \gamma\delta=-s.$$

Let $\alpha+\beta=l$ and $\gamma+\delta=m$.

$\therefore$ $x^4+px^3+qx^2+rx+s=(x^2-lx-1)\,(x^2-mx-s)$.

Comparing the coefficients, we get

(1) $l+m=-p$.

(2) $lm-1-s=q$. $\therefore$ $lm=q+s+1$.

(3) $m+sl=r$.

From (1) and (3), we get $(s-1)\,l=r+p$.

$$\therefore\quad l=\frac{r+p}{s-1},\ \therefore\ m=-p-l=-\frac{ps+r}{s-1}\ \text{from (1).}$$

Putting the values of l and m in (2) and simplifying, we get the required conditions as

$$s^3+s^2\,(q-1)+s\,(p^2+pr-2q-1)+(1+q+pr+r^2)=0.$$

(f) *The roots of equation* $x^4-6x^3+18x^2-30x+25=0$ *are of the form* $\alpha+i\beta$ *and* $\beta+i\alpha$ *where* α *and* β *are real. Find all the roots.* **(Punjab 58)**

If $\alpha+i\beta$ is a root then $\alpha-i\beta$ is also a root.

Similarly if $\beta+i\alpha$ is a root then $\beta-i\alpha$ is also a root.

Sum of the roots$=(\alpha+\beta)+(\beta+\alpha)=6$, $\therefore$ $\alpha+\beta=3$.

Product of all$=(\alpha^2+\beta^2)\,(\beta^2+\alpha^2)=25$, $\therefore$ $\alpha^2+\beta^2=5$.

$\therefore$ $2\alpha\beta=(\alpha+\beta)^2-(\alpha^2+\beta^2)=9-5=4$ or $\alpha\beta=2$

and we have seen that $\alpha+\beta=3$.

$\therefore$ α and β are the roots of t quadratic

$$t^2-3t+2=0 \text{ giving } t=2,\ 1.$$

$$\therefore\quad \alpha=2 \text{ and } \beta=1.$$

Hence the roots are $2\pm i$ and $1\pm 2i$.

11. *Find the conditions that the equation*

$$x^4+px^3+qx^2+rx+s=0,$$

may have its roots in arithmetical progression and hence solve the equation $x^4+2x^3-21x^2-22x+40=0.$

Let the roots be $a-3d,\ a-d,\ a+d,\ a+3d$.

$$\therefore \quad \Sigma\,\alpha=4a=-p, \quad \therefore \quad a=-\frac{p}{4} \quad \ldots \quad \ldots (1)$$

$$\Sigma\,\alpha\beta=6a^2-10d^2=q \quad \text{or} \quad 6.\frac{p^2}{16}-10d^2=q.$$

$$\therefore \quad \frac{3p^2-8q}{80}=d^2 \ldots \quad \ldots \quad \ldots \quad \ldots (2)$$

$$\Sigma\,\alpha\beta\gamma=4a\,(a^2-5d^2)=-r \quad \ldots \quad \ldots \quad \ldots (3)$$

$$\alpha\beta\gamma\delta=(a^2-d^2)\,(a^2-9d^2)=s \quad \ldots \quad \ldots \quad \ldots (4)$$

The required conditions are obtained by putting the values of a and d^2 in (3) and (4) and they are

$$p^3-4pq+8r=0 \text{ and } (36q-11p^2)\,(4q+p^2)=160s.$$

$$\Sigma\,\alpha=4a=-2. \quad \therefore \quad a=-\tfrac{1}{2}$$

and $\quad \Sigma\,\alpha\beta=6a^2-10d^2=-21, \quad \therefore \quad 6.\tfrac{1}{4}+21=10d^2$

or $\quad d^2=\tfrac{9}{4}, \quad \therefore \quad d=\pm\tfrac{3}{2},$

and hence the roots are $-5,\ -2,\ 1,\ 4$.

12. (a) *Find the conditions that the biquadratic*

$$x^4+px^3+qx^2+rx+s=0$$

may have two pairs of equal roots and hence solve the equation

$$x^4-14x^3+73x^2-168x+144=0.$$

Let the two pairs be $\alpha,\ \beta$ and $\alpha,\ \beta$.

$$\therefore \quad \Sigma\,\alpha=2\,(\alpha+\beta)=-p, \quad \therefore \quad \alpha+\beta=-p/2.$$

Also product of the roots$=\alpha^2\beta^2=s, \quad \therefore \quad \alpha\beta=\sqrt{s}.$

Therefore α and β are the roots of $x^2+\dfrac{p}{2}x+\sqrt{s}$.

$$\therefore \quad x^4+px^3+qx^2+rx+s=\left(x^2+\frac{p}{2}x+\sqrt{s}\right)^2$$

$$=x^4+px^3+x^2\left(\frac{p^2}{4}+2\sqrt{s}\right)+px\sqrt{s}+s.$$

Comparing the coefficients, we get

$$q=\frac{p^2}{4}+2\sqrt{s} \quad \text{or} \quad (4q-p^2)^2=(8\sqrt{s})^2=64s$$

and $$r=p\sqrt{s} \quad \text{or} \quad r^2=p^2s$$

are the required conditions.

Just as above the given equation should be equal to

$$(x^2-\tfrac{14}{2}x+\sqrt{144})^2, \quad \text{or} \quad x^2-7x+12=0.$$

$\therefore$ $x=3, 4$. Hence its roots are 3, 4, 3, 4.

Ex. 12 (b) *If* $z^4+6Hz^2+4Gz+(a_0{}^2I-3H^2)=0$ *has two pairs of equal roots, prove that* $G=0$ *and* $a_0{}^2I=12H^2$.

Proceed as above. **(Pb. 26)**

13. *Find the condition that the roots of the equation* $x^4+px^3+qx^2+rx+s=0$ *be connected by the relation* $\alpha\beta+\gamma\delta=0$.

$$\alpha\beta+\gamma\delta=0 \text{ (given)} \ldots \quad \ldots \quad \ldots \quad \ldots(1)$$

Let $\alpha\beta=k$ then $\gamma\delta=-k$,

If $\alpha+\beta=l$ and $\gamma+\delta=m$.

then $(x^4+px^3+qx^2+rx+s)\equiv(x^2-lx+k)(x^2-mx-k)$.

Comparing the coefficients, we get

$$l+m=-p,\ lm=q,\ k(l-m)=r; \ \therefore\ l-m=r/k \text{ and } -k^2=s.$$

In order to find the condition we have to eliminate l, m and k between the above four relations.

$$(l-m)^2=(l+m)^2-4lm.$$

$$\therefore \quad \frac{r^2}{k^2}=p^2-4q \quad \text{or} \quad r^2=-s(p^2-4q)$$

or $$p^2s+r^2=4qs \text{ is the required condition.}$$

14. *If* $x+ay+a^2z=a^3$, $x+by+b^2z=b^3$, $x+cy+c^2z=c^3$, *find the values of* x, y *and* z.

From the given equations we can easily observe that a, b, c satisfy the cubic $t^3=t^2z+ty+x$.

or $t^3-t^2z-ty-x=0$ has for its roots $a, b, c.,$

$$\therefore \quad \Sigma a=a+b+c=z,$$

$$\Sigma ab=ab+bc+ca=-y,$$

$$\Sigma abc=abc=x.$$

Hence the values of x, y and z.

15. *Show that all the roots of the equation*

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots\ldots+p_{n-1}x+p_n=0$$

can be obtained when they are in arithmetical progression.

(Rajputana 55, Agra 29)

The roots of the given equation being in A. P. may be taken to be $a, a+d, a+2d\ldots\ldots a+(n-1)d$. We have to find the values of a and d.

$$\Sigma\alpha=na+\{1+2+3+4+\ldots\ldots(n-1)\}\,d=-p_1$$

or
$$na+\frac{(n-1).n}{2}\,d=-p_1 \quad \ldots \quad \ldots \quad \ldots(1)$$

Again $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2\,(\alpha\beta+\beta\gamma+\gamma\alpha)$

i.e. $\Sigma\alpha^2=(\Sigma\alpha)^2-2\,(\Sigma\alpha\beta)$.

The above relation holds good for any number of roots.

or $a^2+(a+d)^2+(a+2d)^2+\ldots\{a+(n-1)d\}^2=p_1^2-2p_2$

or $na^2+2ad\,\{1+2+3+\ldots+(n-1)\}+d^2\{1^2+2^2+\ldots$
$+(n-1)^2\}=p_1^2-2p_2$

Now $\Sigma n=\dfrac{n\,(n+1)}{2}$ and $\Sigma n^2=\dfrac{n\,(n+1)\,(2n+1)}{6}$.

Putting $n=n-1$ in the above formulae, we get

$$na^2+2ad\,\frac{n\,(n-1)}{2}+d^2\,\frac{n\,(n-1)\,(2n-1)}{6}=p_1^2-2p_2 \quad \ldots.(2)$$

Now if we square (1) and subtract it from n times (2), a will be eliminated and we can find d^2. Having found d we can find a from (1). When a and d are known all the roots are known.

CHAPTER III

TRANSFORMATION OF EQUATIONS

§ 1. In order to solve a given equation $f(x)=0$ we want to transform it into another whose roots bear a certain assigned relation with those of the given equation and whose solution may be more convenient than that of the given equation. Having solved the transformed equation we can find the roots of the given equation from the given relation between the roots of the two equations.

§ 2. To transform an equation into another whose roots are the reciprocals of the roots of the given equation.

Let $f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_{n-1}x+a_n=0$ be the given equation. If x be a root of the given equation and y that of the transformed equation, then $y=\frac{1}{x}$ or $x=\frac{1}{y}$.

Hence the transformed equation is obtained by putting $x=\frac{1}{y}$ in $f(x)=0$ and is therefore $f\left(\frac{1}{y}\right)=0$,

i.e. $$a_0\cdot\frac{1}{y^n}+a_1\cdot\frac{1}{y^{n-1}}+a_2\cdot\frac{1}{y^{n-2}}+\dots a_{n-1}\frac{1}{y}+a_n=0,$$

or $$a_ny^n+a_{n-1}y^{n-1}+\dots+a_2y^2+a_1y+a_0=0.$$

Rule— *If the given equation be complete (if not, it may be made to take that form) then the above transformation is effected if we take the last coefficient to be the 1st, last but one to be the 2nd and so on.*

§ 2·1. Reciprocal equations.

All those equations which remain unchanged when x is replaced by $1/x$ are called reciprocal equations. These are of two types (i) :— those in which the coefficients of terms, equidistant from the beginning and the end, are equal and of the same sign, and (ii), those in which these coefficients are equal but of opposite signs, *e.g.*,

$x^4+6x^3+8x^2+6x+1=0$ is of the first type,

and $x^5-4x^4+7x^3-7x^2+4x-1=0$ is of the 2nd type.

Thus we conclude that if α is a root of the reciprocal equation then $1/\alpha$ must also be its root. Hence the roots of a reciprocal equation occur in pairs of α, $1/\alpha$, β, $1/\beta$, γ, $1/\gamma$ and so on.

In case the equation be of odd degree then it will be seen that one of its roots must be either $+1$ or -1. In case the equation be of even degree and of 2nd type then it will be seen that x^2-1 will always be its factor, *e.g.*,

$$6x^6+5x^5-44x^4+44x^2-5x-6=0$$

or $$6(x^6-1)+5x(x^4-1)-44x^2(x^2-1)=0$$

or $$(x^2-1)[6(x^4+x^2+1)+5x(x^2+1)-44x^2]=0$$

or $$(x^2-1)(6x^4+5x^3-38x^2+5x+6]=0$$

Above shows that (x^2-1) is a factor of the given equation. The equation $6x^4+5x^3-38x^2+5x+6=0$ is of even degree and of the first type, *i.e.* the coefficients of terms equidistant from the beginning and the end are equal, and is called in the standard form to which all the reciprocal equations can be reduced.

Ex. 1. *Find the condition that the roots of the equation* $x^3-px^2+qx-r=0$ *be in harmonical progression.*

(Delhi Hon's 52, Agra 34, 37, 45, Pb. 36, 51)

Hence solve the equation $6x^3-11x^2-3x+2=0$. **(Pb. 54)**

Since the roots of $x^3-px^2+qx-r=0$ are in H. P., hence the roots of the equation $-rx^3+qx^2-px+1=0$ or of

$rx^3 - qx^2 + px - 1 = 0$ are in A. P. and let us take them to be $a-d$, a and $a+d$.

∴ $\Sigma\alpha = 3a = \frac{q}{r}$; ∴ $a = \frac{q}{3r}$; but a is the root of the transformed equation.

∵ $r\left(\frac{q}{3r}\right)^3 - q.\left(\frac{q}{3r}\right)^2 + p.\left(\frac{q}{3r}\right) - 1 = 0$ and hence the required condition is $27r^2 - 9pqr + 2q^3 = 0$.

The transformed equation of $6x^3 - 11x^2 - 3x + 2 = 0$ is $2x^3 - 3x^2 - 11x + 6 = 0$ and its roots being in A. P. may be taken as $a-d$, a, $a+d$.

$$\therefore \quad \Sigma\alpha = 3a = 3/2 \quad \text{or} \quad a = 1/2,$$

and $\Sigma\alpha\beta\gamma = a(a^2 - d^2) = -6/2$ or $1/2.(1/4 - d^2) = -3$ or $d^2 = 25/4$ ∴ $d = \frac{5}{2}$ and hence the roots of transformed equation are -2, $1/2$, 3. Therefore the roots of the given equation are $-1/2$, 2, $1/3$.

Note—See for alternative method Q. 5, P. 25.

Ex 2. *Find the conditions that the roots of the equation* $x^4 + px^3 + qx^2 + rx + s = 0$ *be in harmonical progression*

Take the transformed equation and find the conditions that its roots may be in A. P., just as in Q. 11. P. 31.

Required conditions are $r^3 = 4qrs - 8ps^2$

and $25r^2s = q(r-s)(11sp - qr)$.

§ 3. To transform an equation into another whose roots are the roots of the given equation with sign changed.

If y be the root of the transformed equation, then $y = -x$ or $x = -y$. Hence the transformed equation is obtained by putting $x = -y$ in $f(x) = 0$ and is therefore $f(-y) = 0$ which takes the form

$$a_0y^n - a_1y^{n-1} + a_2y^{n-2} - \ldots + (-1)^n a_n = 0.$$

Rule :—*If the given equatioa be complete (if not it may be made to take that form), then the above transformation is effected by changing the sign of 2nd, 4th, 6th i.e. all even terms i.e. by changing the sign of every alternate term beginning from the 2nd.*

Ex. 1. *Form equations whose roots are the roots of the following equations with their sign changed :—*

(i) $x^3-5x^2-7x+3=0.$

(ii) $x^5-x^2+x-3=0.$

(iii) $-4x^3+2x^2-3x-5=0.$

Ans. (i) $x^3+5x^2-7x-3=0.$

(ii) The given equation being not complete may be written as

$$x^5+0x^4+0x^3-x^2+x-3=0$$

and the transformed equation therefore is

$$x^5-0x^4+0x^3+x^2+x+3=0$$

or

$$x^5+x^2+x+3=0.$$

(iii) $-4x^3-2x^2-3x+5=0$ or $4x^3+2x^2+3x-5=0.$

Ex. 2. *Solve the equation* $x^5-3x^4-5x^3+15x^2+4x-12=0$ *which has got some pairs of roots equal in magnitude but opposite in sign.*

The number of pairs that the equation of 5th degree can have is two. Since the pairs are equal in magnitude but of opposite sign and as such if α, β are two of its roots then $-\alpha$, $-\beta$ must also be roots of this equation which means α, β will also be the roots of

$$x^5+3x^4-5x^3-15x^2+4x+12=0$$

i.e., the equation in which the sign of every alternate term is changed. Thus α and β are the common roots of the given equation and the above equation and are obtained by finding the H.C.F. of the two which is

$$x^4-5x^2+4=0 \quad \text{or} \quad (x^2-1)(x^2-4)=0$$

and dividing the given equation by x^4-5x^2+4 the remaining factor is found to be $x-3$. Hence the roots of the given equation are ± 1, ± 2, and 3.

§ 4. **To transform a given equation into another whose roots are the roots of the given equation multiplied by a given number m.**

If y be a root of the given equation, then $y=mx$ or $x=\frac{y}{m}$.

Hence the transformed equation is obtained by putting $x=\frac{y}{m}$ in $f(x)=0$ and is therefore $f\left(\frac{y}{m}\right)=0$.

i.e. $$a_0\frac{y^n}{m^n}+a_1\frac{y^{n-1}}{m^{n-1}}+a_2\frac{y^{n-2}}{m^{n-2}}+\ldots+a_n=0$$

or $$a_0y^n+a_1my^{n-1}+a_2m^2y^{n-2}+\ldots+m^na_n=0.$$

Rule :—*If the given equation be complete (if not, it may be made complete), then the above transformation is effected by multiplying the successive terms beginning from the 2nd by* $m, m^2, m^3\ldots m^n$ *respectively.*

Note. (i) The above transformation is very useful when we are dealing with equations with fractional coefficients. We can get rid of fractional coefficients by multiplying the roots of the given equation by the L. C. M. of the denominators of the fractional coefficients. Similarly if the coefficient of leading term be not unity but k and we want to make it unity, then it can be done so by multiplying the roots of the given equation by k.

Note. (ii) If we have to divide the roots of the given equation by m, we say that we have to multiply its roots by $\frac{1}{m}$.

Ex. 1. *Transform the following equations into one in which the leading coefficient be 1.*

(i) $2x^4-5x^2+3x-4=0.$

(ii) $5x^3+3x^2+2x-4=0.$

(i) The given equation being not complete may be made complete by writing it in the form $2x^4+0x^3-5x^2+3x-4=0$. In order to make the leading coefficient 1, multiply the roots of this equation by 2, and the transformed equation is

$$2x^4+2\,(0x^3)-2^2\,(5x^2)+2^3\,(3x)-2^4\,(4)=0$$

or
$$x^4-10x^2+12x-32=0$$

(ii) $x^3+3x^2+10x-100=0.$

Ex. 2. *Remove the fractional coefficients from the equation*

(i) $x^3-\frac{5}{2}x^2-\frac{7}{18}x+\frac{1}{108}=0.$

The L. C. M. of the denominators of the fractional coefficients being 108, it will be very difficult to multiply the roots by 108 which would be very tedious and our purpose can be served by multiplying the roots by a smaller number say *m*. Split the denominators into smallest factors and multiply the roots by *m*. The transformed equation is

$$x^3-m.\frac{5}{2^1.3^0}x^2-m^2\frac{7}{2^1.3^2}x+m^3\frac{1}{2^2.3^3}=0.$$

It is easily clear from the above that the fractional coefficients can be removed if we choose $m=2\times 3$ and hence the transformed equation is

$$x^3-15x^2-14x+2=0.$$

The following method may be used in finding the value of *m*. The exponents of 2 in order are (1, 1, 2).

Divide them by 1, 2, 3 respectively and we get the quotients as 1, $\frac{1}{2}$, $\frac{2}{3}$. The least integer not less than any of these quotients is 1 and hence 2^1 will be a factor of m.

Similarly the exponents of 3 in order are (0, 2, 3). Divide them by 1, 2, 3 respectively and we get the quotients as 0, 1, 1. The least integer not less than any of these quotients is 1 and hence 3^1 will be a factor of m. Thus m should be $2^1 \times 3^1$ *i. e.* 6.

(ii) $x^4 - \frac{x^3}{2} + \frac{x^2}{9} - \frac{x}{8} + \frac{1}{12} = 0.$

Proceeding as above we have to multiply the roots by $2^1 \times 3^1$ or by 6.

Note. In this case we shall have to divide the successive exponents of 2 and 3 by 1, 2, 3 and 4 respectively.

(iii) $x^4 - \frac{x^3}{6} - \frac{x^2}{60} + \frac{x}{200} + \frac{1}{2250} = 0.$

If we multiply the roots by m, then the transformed equation is

$$x^4 - m\frac{x^3}{2^1.3^1.5^0} - m^2\frac{x^2}{2^2.3^1.5^1} + m^3\frac{x}{2^3.3^0.5^2} + m^4\frac{1}{2^1.3^2.5^3}.$$

Proceeding as above or otherwise, it is quite clear that m should be chosen equal to $2 \times 3 \times 5$ *i. e.* 30 in order to remove the fractional coefficients and the transformed equation is $x^4 - 5x^3 - 15x^2 + 135x + 360 = 0.$

(iv) $x^4 + \frac{3}{10}x^2 + \frac{13}{25}x + \frac{77}{1000} = 0.$ **(Pb. 45)**

Multiplying by 10, $x^4 + 30x^2 + 520x + 770 = 0.$

(v) $x^4 - \frac{5}{6}x^3 + \frac{5}{12}x^2 - \frac{13}{900} = 0.$

Make the equation complete and multiply the roots by m.

$$x^4-5.m.\frac{x^3}{2^1.3^1.5^0}+5.m^2.\frac{x^2}{2^2.3^1.5^0}$$
$$+0.m^3.x-m^4\frac{13}{2^2.3^2.5^2}.$$

It is clear that $m=2^1.3^1.5^1$ and this value of m removes all the fractional coefficients.

Ans. $x^4-25x^3+375x^2-11700=0$.

(v) *The roots of the equation* $81x^3-18x^2-36x+8=0$ *are in H. P. Transform it into another with integral coefficients and unity for the coefficient of the first term, so that the roots of the transformed equation be in A.P.*

First find the equation whose roots are the reciprocals of the roots of the given equation and multiply the roots by 2 etc.

Ans. $x^3-9x^2-9x+81=0$.

§ 5. To transform a given equation into another whose roots are the roots of the given equation diminished (or increased) by a constant h.

Let $f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0\,..(1)$

If y be a root of the given equation, then $y=x-h$ or $x=y+h$. Hence the transformed equation is obtained by putting $x=y+h$ in $f(x)=0$ and is therefore $f(y+h)=0$

or $\quad a_0(y+h)^n+a_1(y+h)^{n-1}+a_2(y+h)^{n-2}+\ldots+a_n=0.$

The simplification of the above equation will be difficult and let us suppose that this equation when simplified and arranged in descending powers of y takes the form $A_0y^n+A_1y^{n-1}+A_2y^{n-2}+\ldots+A_{n-1}y+A_n=0\ldots\ldots(2)$

The problem is to find $A_0, A_1, A_2 \ldots A_n$. Now $y = x - h$.

$$\therefore \quad A_0 (x-h)^n + A_1 (x-h)^{n-1} + A_2 (x-h)^{n-2} + \ldots$$
$$A_{n-1} (x-h) + A_n = 0$$

or

$$[A_0 (x-h)^{n-1} + A_1 (x-h)^{n-2} + A_2 (x-h)^{n-3} + \ldots + A_{n-1}]$$
$$\times (x-h) + A_n = 0 \quad \ldots (3)$$

The L. H. S. of above is identical with L. H. S. of (1) and hence if $f(x)$ be divided by $(x-h)$, then the remainder is the value of A_n and the quotient is

$$A_0 (x-h)^{n-1} + A_1 (x-h)^{n-2} + \ldots + A_{n-1}$$

and the quotient when again divided by $(x-h)$, leaves the remainder A_{n-1}. If we continue the above process, then we shall find $A_n, A_{n-1} \ldots A_2, A_1$ and the last quotient A_0 is clearly equal to a_0.

Rule :—*In order to find the successive coefficients of the transformed equation we have to divide the given complete equation by $(x-h)$, the quotient again by $x-h$, and so on. The successive remainders that are left in the above procedure of division are the required coefficients, the first coefficient being the same as that of the given equation.*

Note :—In case we are to increase the roots by h *i.e.* diminish by $-h$, then the division is to be done by $(x+h)$. We shall use **synthetic division as** explained on **P. 6.**

Ex. *Diminish by 2 the roots of the equation*

$$x^5 - 3x^4 - 2x^3 + 15x^2 + 20x - 15 = 0. \qquad \textbf{(Pb. 37)}$$

Let the transformed equation be

$$A_0 x^5 + A_1 x^4 + A_2 x^3 + A_3 x^2 + A_4 x + A_5 = 0 \text{ where } A_0 = a_0 = 1.$$

In order to find these A's we have to divide the given equation by $x-2$ by the process of synthetic division and the remainder will be A_5. The quotient when again divided by $(x-2)$ will leave the remainder which will be the value of A_4 and so on. The process is exhibited below.

$$\begin{array}{l|rrrrrr}
2 & 1 & -3 & -2 & 15 & 20 & -15 \\
 & & 2 & -2 & -8 & 14 & 68 \\
\hline
Q.\ 1 & & -1 & -4 & 7 & 34 & \boxed{53=A_5} \\
 & & 2 & 2 & -4 & 6 & \\
\hline
Q.\ 2 & & 1 & -2 & 3 & \boxed{40=A_4} & \\
 & & 2 & 6 & 8 & & \\
\hline
Q.\ 3 & & 3 & 4 & \boxed{11=A_3} & & \\
 & & 2 & 10 & & & \\
\hline
Q.\ 4 & & 5 & \boxed{14=A_2} & & & \\
 & & 2 & & & & \\
\hline
 & & 7=A_1 & & & &
\end{array}$$

Hence the transformed equation is

$$x^5+7x^4+14x^3+11x^2+40x+53=0.$$

Ex. 2. *Find the equation whose roots are the roots of the equation $4x^5-2x^3+7x-3=0$ each increased by 2.* **(Pb. 1940)**

The equation being not complete may therefore be written in the form $4x^5+0x^4-2x^3+0x^2+7x-3=0$.

Also increasing the roots by 2 means diminishing them by -2 *i.e.* we have to divide the equation by $x-(-2)$ in order to find the successive coefficients of the transformed equation whose first coefficient will be the same as that of the given equation i.*e.* $A_0=a_0=4$.

$$\begin{array}{l|rrrrrr}
-2 & 4 & 0 & -2 & 0 & 7 & -3 \\
 & & -8 & 16 & -28 & 56 & -126 \\
\hline
Q.\ 1 & & -8 & 14 & -28 & 63 & \boxed{-129=A_5} \\
 & & -8 & 32 & -92 & 240 & \\
\hline
Q.\ 2 & & -16 & 46 & -120 & \boxed{303=A_4} & \\
 & & -8 & 48 & -188 & & \\
\hline
Q.\ 3 & & -24 & 94 & \boxed{-308=A_3} & & \\
 & & -8 & 64 & & & \\
\hline
Q.\ 4 & & -32 & \boxed{158=A_2} & & & \\
 & & -8 & & & & \\
\hline
 & & -40=A_1 & & & &
\end{array}$$

Hence the transformed equation is

$$4x^5-40x^4+158x^3-308x^2+303x-129=0.$$

Ex. 3. *Diminish the roots of the equation*

$$5x^3-13x^2-12x+7=0 \text{ by } 23.$$

The usual process will be very difficult if we straight off begin to diminish the roots by 23 because of the long multiplication involved. In such type of questions the diminishing may be done in two stages. We shall first diminish the roots by 20 and then by 3 as shown below.

$$\begin{array}{r|rrrr}
20 & 5 & -13 & -12 & 7 \\
 & & 100 & 1740 & 34560 \\
\cline{5-5}
 & & 87 & 1728 & 34567=A_3 \\
 & & 100 & 3740 & \\
\cline{4-4}
 & & 187 & 5468=A_2 & \\
 & & 100 & & \\
\cline{3-3}
 & & 287=A_1 & &
\end{array}$$

Then the equation whose roots are diminished by 20 is

$$5x^3+287x^2+5468x+34567=0.$$

Diminish again the roots of the above equation by 3 and the resulting equation obtained is

$$5x^3+332x^2+7325x+53689=0.$$

Note—In case we are to diminish the roots by 17, then we should first diminish by 20 and then increase the roots of the transformed equation by 3 *i.e.* diminish by -3.

Ex. 4. *Find the equation whose roots are the roots of*

$$x^5+4x^3-x^2+11=0$$

each diminished by 3. **(Agra 55)**

Ans. $x^5+15x^4+94x^3+305x^2+507x+353=0.$

§ 6. Removal of terms :—

Sometimes we want to get rid of a particular term in the given equation.

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0 \quad \ldots\ldots(1)$$

We diminish the roots of $f(x)=0$ by a constant h and the resulting equation is obtained by putting $y=x-h$ or $x=y+h$ in (1) and is therefore

$$a_0(y+h)^n+a_1(y+h)^{n-1}+a_2(y+h)^{n-2}+\dots$$
$$+a_{n-1}(y+h)+a_n=0$$

or

$$a_0 y^n+(na_0h+a_1)y^{n-1}$$
$$+\left[\frac{n(n-1)}{2!}a_0h^2+(n-1)a_1h+a_2\right]y^{n-2}$$
$$+\left[\frac{n(n-1)(n-2)}{3!}a_0h^3+\frac{(n-1)(n-2)}{2!}a_1h^2\right.$$
$$\left.+(n-2)a_2h+a_3\right]y^{n-3}+\dots$$
$$+[a_0h^n+a_1h^{n-1}+\dots+a_{n-1}h+a_n]=0.$$

Now if we want to remove any term we equate to zero the coefficient of that particular term which is a function of h. The value of h obtained from it will give the required number by which the roots be diminished in order to remove any term. It may be noted here that if we want to remove the 2nd term there will be only one value of h as its coefficient is a linear function of h. In case we desire to remove the 3rd term then we shall have two values of h for its coefficient is quadratic function of h and so on.

Ex. 1. *(a) Show that the same transformation removes both 2nd and 4th terms of the equation,*

$$x^4+16x^3+83x^2+152x+84=0. \qquad \textbf{(Pb. 34)}$$

In case we want to remove the 2nd term we should diminish the roots by h which may be so chosen as to satisfy the relation $na_0h+a_1=0$ *i.e.* $4.1.h+16=0$ *i.e.* $h=-4$. It may be seen that the value of h makes the coefficient of fourth term zero. But we should better diminish the roots by -4 and then show that in the transformed equation the coefficient of fourth term is zero.

$$\begin{array}{r|rrrrr}
-4 & 1 & 16 & 83 & 152 & 84 \\
 & & -4 & -48 & -140 & -48 \\
\hline
 & & 12 & 35 & 12 & 36 \\
 & & -4 & -32 & -12 & \\
\hline
 & & 8 & 3 & 0 \text{ (Note)} & \\
 & & -4 & -16 & & \\
\hline
 & & 4 & -13 & & \\
 & & -4 & & & \\
\hline
 & & 0 & & &
\end{array}$$

Hence the transformed equation is $y^4-13y^2+36=0$ which is wanting in both the 2nd and 4th terms.

or $(y^2-9)(y^2-4)=0$; $\therefore$ $y=3, -3, 2, -2$.

Now $y=x-h=x+4$; $\therefore$ $x=y-4$.

Hence the roots of the given equation are

$(3-4), (-3-4), (2-4), (-2-4)$ *i.e.* $-1, -7, -2, -6$.

(b) Solve the equation $x^4+20x^3+143x^2+430x+462=0$ by removing its 2nd term. **(Pb. 37)**

Proceed as above. When the roots are diminished by -5, both 2nd and 4th terms are removed and the equation becomes $y^4-7y^2+12=0$. **Ans.** $-7, -3, -5, \pm\sqrt{3}$.

(c) Transform the equation $x^4+8x^3+x-5=0$ into one in which the 2nd term is wanting. **(Pb. 48)**

Now $na_0h+a_1=0$ gives $h=-2$. Diminish the roots by -2 after making the given equation complete.

Ans. $x^4-24x^2+65x-55=0$.

Ex. 2. *Transform the equation $x^4-4x^3-18x^2-3x-2=0$ into one which shall want the third term.* **(Pb. 44)**

In case we want to remove the third term then the value of h is obtained by $\frac{n(n-1)}{2!}a_0h^2+(n-1)a_1h+a_2=0$, where $n=4$, $a_0=1$, $a_1=-4$ and $a_2=-18$.

$$\therefore \quad h^2-2h-3=0\,; \quad h=-1,\ 3.$$

If we diminish the roots by -1, the transformed equation is $x^4-8x^3+17x-12=0$ and if h be taken equal to 3, then the transformed equation is

$$x^4+8x^3-111x-196=0.$$

Ex. 3. *(a) Reduce the cubic* $2x^3-3x^2+6x-1=0$ *to the form* $z^3+3Hz+G=0$.

First multiply the roots by 2 in order to make the coefficient of leading term unity and the transformed equation is $y^3-3y^2+12y-4=0$. Now diminish the roots by 1 etc. $z^3+9z+6=0$.

(b) Remove the 2nd term of the equation $x^3+6x^2+12x-19=0$ *and hence solve it completely.*

$$na_0h+a_1=0 \text{ gives } 3h+6=0 \quad i.e. \quad h=-2.$$

Diminish the roots by -2 and transformed equation is $y^3-27=0$ whose roots are $3, 3\omega. 3\omega^2$ *i. e.*

$$y=3,\ 3\left(\frac{-1+i\sqrt{3}}{2}\right),\ 3\left(\frac{-1-i\sqrt{3}}{2}\right).$$

But $y=x-h=x+2 \quad \therefore \quad x=y-2$ and hence the roots of the given equation are $1,\ \dfrac{-7+i.3\sqrt{3}}{2},\ \dfrac{-7-i.3\sqrt{3}}{2}$.

(c) Solve as above the cubic $x^3-12x^2+48x-72=0$.

$6,\ 3\pm i\sqrt{3}$. **Ans.**

§ 7. Binomial Coefficients :

The successive binomial coefficients in the expansion of $(1+x)^n$ are

$$1,\ n,\ \frac{n(n-1)}{2\,!},\ \frac{n(n-1)(n-2)}{3\,!}\ldots n,\ 1$$

and that the coefficients equidistant from the beginning and the end are equal.

$n=1$ coeffs. are 1 1
$n=2$ „ 1 2 1
$n=3$ „ 1 3 3 1
$n=4$ „ 1 4 6 4 1

We have so far written the polynomial $f(x)$ as

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+a_3x^{n-3}+\dots+a_{n-1}x+a_n=0.$$

It will be of great advantage if we take our polynomial which will have for its coefficients the letters $a_0, a_1, a_2, a_3 \dots\dots a_n$ multiplied repectively by the binomial coefficients in the expansion of $(1+x)^n$ *i.e.* by

$$1,\ n,\ \frac{n(n-1)}{2!},\ \frac{n(n-1)(n-2)}{3!}\dots.n,\ 1.$$

Let us denote this polynomial by ϕ_n. Thus

$$\phi_n=a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{2!}a_2x^{n-2}+\dots na_{n-1}x+a_n.$$

Putting $n=n-1$, we get

$$\phi_{n-1}=a_0x^{n-1}+(n-1)a_1x^{n-2}+\dots+(n-1)a_{n-2}x+a_{n-1}.$$

Putting $n=4$, 3, 2, 1, we get

$$\phi_4=a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4,$$
$$\phi_3=a_0x^3+3a_1x^2+3a_2x+a_3,$$
$$\phi_2=a_0x^2+2a_1x+a_2,$$
$$\phi_1=a_0x+a_1,$$
$$\phi_0=a_0.$$

It can easily be seen that the differential coefficient of ϕ_n is n times ϕ_{n-1}, *i. e.*, $\phi'n=n.\phi_{n-1}$, $\phi'_4=4.\phi_3$, $\phi'_3=3.\phi_2$ and $\phi'_2=2\phi_1$ and so on.

Lot of our botheration of simplification will be saved if we are to diminish the roots of a given equation if it be ith binomial coefficients as shown below :

$$f(x)=a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{2!}a_2x^{n-2}+\ldots$$
$$+na_{n-1}x+a_n=0.$$

If we diminish the roots by h *i.e.* $y=x-h$ or $x=y+h$, get

$$a_0(y+h)^n+na_1(y+h)^{n-1}+\frac{n(n-1)}{2!}a_2(y+h)^{n-2}+\ldots$$
$$+na_{n-1}(y+h)+a_n=0.$$

Simplifying, we get

$$a_0y^n+(na_0h+na_1)y^{n-1}$$
$$+\left(\frac{n(n-1)}{2!}a_0h^2+n(n-1)a_1h+\frac{n(n-1)a_2}{2!}\right)y^{n-2}+\ldots$$

ich can be put in the form

$$a_0y^n+n(a_0h+a_1)y^{n-1}+\frac{n(n-1)}{2!}[a_0h^2+2a_1h+a_2]y^{n-2}+\ldots=0$$

Then the transformed equation is

$$A_0y^n+nA_1y^{n-1}+\frac{n(n-1)}{2!}A_2y^{n-2}+\ldots+nA_{n-1}y+A_n=0$$

ere $A_0=a_0$ and $A_1, A_2, A_3 \ldots\ldots. A_n$ are the values of $\phi_2, \phi_3 \ldots. \phi_n$ when x is replaced by h.

Hence if we have a cubic equation with binomial fficients *i.e.* $a_0x^3+3a_1x^2+3a_2x+a_3=0$ and its roots be ninished by h, then we can at once write down the transned equation as

$$A_0y^3+3A_1y^2+3A_2y+A_3=0,$$

ere $A_0=a_0$, $A_1=a_0h+a_1$, $A_2=a_0h^2+2a_1h+a_2$ and $=a_0h^3+3a_1h^2+3a_2h+a_3$.

Similarly, if we have a biquadratic with binomial fficients *i. e.* $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ then as a result of diminishing its roots by *h*, the transformed equation will be

$$A_0y^4+4A_1y^3+6A_2y^2+4A_3y+A_4=0$$

where A_0, A_1, A_2, A_3 have the same values as above and

$$A_4=a_0h^4+4a_1h^3+6a_2h^2+4a_3h+a_4.$$

§ 8. Important symbols to be committed to memory.

When the equation is written with binomial coefficients then we shall hencefourth use the following symbols.

$$H=a_0a_2-a_1^2=\begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}.$$

$$G=a_0^2a_3-3a_0a_1a_2+2a_1^3=\begin{vmatrix} 0 & a_0 & a_1 \\ a_0 & a_1 & a_2 \\ 2a_1 & a_2 & a_3 \end{vmatrix}.$$

$$I=a_0a_4-4a_1a_3+3a_2^2.$$

$$J=a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3=\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}$$

a_0 stands for coeff. of x^3, a_1 for one-third the coefficient of x^2, a_2 for one-third the coefficient of x and a_3 for constant term for a cubic with binomial coefficients and similarly for biquadratic with binomial coefficients.

It may be noted that all the symbols have the following peculiarities :—

(1) Sum of all the +ive coefficients=sum of the −ive coefficients.

(2) Sum of the suffixes of the letters '*a*' in each term is same.

$a_1^3=a_1a_1a_1$ and sum of the suffixes$=1+1+1=3$ etc.

(3) Sum of the exponents of the letters 'a' in each term is same.

There also exists a very important relation between these symbols, *i.e.*

$$G^2+4H^3=a_0^2\,(HI-a_0J).$$

The students may verify for themselves.

§ 9. Reduce the cubic with binomial coefficients i.e. $a_0x^3+3a_1x^2+3a_2x+a_3=0$ to the form in which 2nd term may be wanting and the coefficient of the leading term be unity and all other coefficients integral. Also write down the roots of the transformed equation if α, β, γ be the roots of the given equation. **(Pb. 55)**

$$f(x)\equiv a_0x^3+3a_1x^2+3a_2x+a_3=0 \quad \ldots \quad \ldots(1)$$

Its roots are α, β, γ.

In order to remove the 2nd term let us diminish the roots of (1) by h ; then the transformed equation is

$$A_0y^3+3A_1y^2+3A_2y+A_3=0.$$

If 2nd term is to be removed, then $A_1=a_0h+a_1=0$.

$$\therefore \quad h=\frac{-a_1}{a_0}.$$

$$\therefore \quad A_2=a_0h^2+2a_1h+a_2=a_0\left(\frac{-a_1}{a_0}\right)^2+2a_1\left(\frac{-a_1}{a_0}\right)+a_2$$

$$=\frac{a_0a_2-a_1^2}{a_0}=\frac{H}{a_0}.$$

$$A_3=a_0h^3+3a_1h^2+3a_2h+a_3$$

$$=a_0\left(\frac{-a_1}{a_0}\right)^3+3a_1\left(\frac{-a_1}{a_0}\right)^2+3a_2\left(\frac{-a_1}{a_0}\right)+a_3$$

$$=\frac{a_0^2a_3-3a_0a_1a_2+2a_1^3}{a_0^2}=\frac{G}{a_0^2}.$$

and $\quad A_0=a_0.$

Hence the transformed equation is

$$a_0 y^3+3\frac{H}{a_0}y+\frac{G}{a_0^2}=0$$

or

$$y^3+3\frac{H}{a_0^2}y+\frac{G}{a_0^3}=0 \quad \ldots \quad \ldots \quad \ldots (2)$$

The roots of (2) are

$$\alpha-h,\ \beta-h,\ \gamma-h$$

or

$$\alpha+\frac{a_1}{a_0},\ \beta+\frac{a_1}{a_0},\ \gamma+\frac{a_1}{a_0},\quad \because\ h=-\frac{a_1}{a_0}.$$

i.e.

$$\frac{a_0\alpha+a_1}{a_0},\ \frac{a_0\beta+a_1}{a_0},\ \frac{a_0\gamma+a_1}{a_0}.$$

Now we want the coefficients in (2) should be integral and hence we should multiply its roots by a_0 and the transformed equation is

$$z^3+0.a_0z^2+a_0^2.\frac{3H}{a_0^2}z+a_0^3.\frac{G}{a_0^3}=0. \qquad \text{(§ 4 P. 37)}$$

i.e.

$$z^3+3Hz+G=0 \ldots \quad \ldots \quad \ldots \quad \ldots (3)$$

is the required form and its roots are

$$a_0.\left(\frac{a_0\alpha+a_1}{a_0}\right),\ a_0.\left(\frac{a_0\beta+a_1}{a_0}\right),\ a_0.\left(\frac{a_0\gamma+a_1}{a_0}\right)$$

i.e.

$$a_0\alpha+a_1,\ a_0\beta+a_1,\ a_0\gamma+a_1.$$

Hence $z=a_0y=a_0(x-h)=a_0\left(x+\frac{a_1}{a_0}\right)=a_0x+a_1.$

If we put $z=a_0x+a_1$ in (3) and simplify, it will be equal to original cubic multiplied by a_0^2 i.e. $a_0^2 f(x)$.

Again $\alpha+\beta+\gamma=-\frac{3a_1}{a_0}$, $\therefore\ -\frac{\alpha+\beta+\gamma}{3}=\frac{a_1}{a_0}$ .....(4)

Now $z^3+3Hz+G=0$ has for its roots

$$a_0\alpha+a_1,\ a_0\beta+a_1,\ a_0\gamma+a_1$$

$$a_0\left(\alpha+\frac{a_1}{a_0}\right),\ a_0\left(\beta+\frac{a_1}{a_0}\right),\ a_0\left(\gamma+\frac{a_1}{a_0}\right)$$

$$a_0\left(\alpha-\frac{\alpha+\beta+\gamma}{3}\right),\ a_0\left(\beta-\frac{\alpha+\beta+\gamma}{3}\right),\ a_0\left(\gamma-\frac{\alpha+\beta+\gamma}{3}\right)$$

$$\frac{a_0}{3}(2\alpha-\beta-\gamma),\ \frac{a_0}{3}(2\beta-\gamma-\alpha),\ \frac{a_0}{3}(2\gamma-\alpha-\beta) \qquad \ldots(5)$$

Ex. 1. *If* $\alpha, \beta, \gamma,$ *be the roots of*

$$a_0x^3+3a_1x^2+3a_2x+a_3=0,$$

ien write down the value of $\Sigma\,(2\alpha-\beta-\gamma)\,(2\beta-\gamma-\alpha)$ *and* $2\alpha-\beta-\gamma)\,(2\beta-\gamma-\alpha)\,(2\gamma-\alpha-\beta)$. **(Agra 41)**

Let the equation be reduced to the form

$$z^3+3Hz+G=0 \text{ where } z=a_0x+a_1$$

nd its roots are $\frac{a_0}{3}(2\alpha-\beta-\gamma)$ etc.

Sum of the products of the roots of the transformed quation taken two at a time $=3H$

$$\frac{a_0^2}{9}\Sigma\,(2\alpha-\beta-\gamma)\,(2\beta-\gamma-\alpha)=3H.$$

$$\therefore\quad (2\alpha-\beta-\gamma)\,(2\beta-\gamma-\alpha)=\frac{27H}{a_0^2}=\frac{27}{a_0^2}(a_0a_2-a_1^2).$$

Similarly $\frac{a_0^3}{27}(2\alpha-\beta-\gamma)\,(2\beta-\gamma-\alpha)\,(2\gamma-\alpha-\beta)=$ product the roots of the equation taken all at a time $=-G$.

$$\therefore\quad (2\alpha-\beta-\gamma)\,(2\beta-\gamma-\alpha)\,(2\gamma-\alpha-\beta)=-\frac{27G}{a_0^3}$$

$$=-\frac{27}{a_0^3}(a_0^2a_3-3a_0a_1a_2+2a_1^3).$$

§ 10. Reduce the biquadratic with binomial coefficients i. e. $\mathbf{a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0}$ **to the form**

in which 2nd term may be wanting and coefficients of leading term be unity and all other coefficients to be integral. Also write down the roots of the transformed equation, if α, β, γ be the roots of the given equation.

$$f(x)=a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0 \quad \ldots \quad \ldots (1)$$

Its roots are α, β, γ, δ.

In order to remove the 2nd term, let us diminish the roots of (1) by h; then the transformed equation is

$$A_0y^4+4A_1y^3+6A_2y^2+4A_3y+A_4=0.$$

If 2nd term is to be removed, then $A_1=a_0h+a_1=0$;

$$\therefore \quad h=-\frac{a_1}{a_0}.$$

For this value of h, $A_2=\dfrac{H}{a_0}$, $A_3=\dfrac{G}{a_0^2}$ as in § 9

and
$$A_4=a_0h^4+4a_1h^3+6a_2h^2+4a_3h+a_4$$
$$=\frac{a_0^3a_4-4a_0^2a_1a_3+6a_0a_1^2a_2-3a_1^4}{a_0^3}$$
$$=\frac{a_0^2I-3H^2}{a_0^3} \text{ (students may verify)}$$

and
$$A_0=a_0.$$

Hence the transformed equation is

$$a_0y^4+\frac{6H}{a_0}y^2+\frac{4G}{a_0^2}y+\frac{a_0^2I-3H^2}{a_0^3}=0$$

or
$$y^4+\frac{6H}{a_0^2}y^2+\frac{4G}{a_0^3}y+\frac{a_0^2I-3H^2}{a_0^4}=0.$$

The roots of (2) are $\alpha-h$, $\beta-h$, $\gamma-h$, $\delta-h$

or
$$\alpha+\frac{a_1}{a_0} \text{ etc, or } \frac{a_0\alpha+a_1}{a_0} \text{ etc.}$$

Now we want that the coefficients in (2) should be integral and hence we should multiply its roots by a_0 and the transformed equation is (§ 4, P. 37)

$$z^4+0a_0z^3+a_0^2\,\frac{6H}{a_0^2}\,z^2+a_0^3.\frac{4G}{a_0^3}\,z+a_0^4.\frac{a_0^2I-3H^2}{a_0^4}=0,$$

i.e.
$$z^4+6Hz^2+4Gz+(a_0^2I-3H^2)=0 \qquad \ldots \quad \ldots(3)$$

is the required form and its roots are

$$a_0\left(\frac{a_0\alpha+a_1}{a_0}\right) \text{ etc. } i.e. \ (a_0\alpha+a_1) \text{ etc.}$$

Hence $z=a_0y=a_0(x-h)=a_0\left(x+\frac{a_1}{a_0}\right)=a_0x+a_1.$

If we put $z=a_0x+a_1$ in (3) and simplify, it will be equal to original biquadratic multiplied by a_0^3, i.e. $a_0^3 f(x)$.

Again $\alpha+\beta+\gamma+\delta=-\frac{4a_1}{a_0}$; $\therefore$ $-\frac{\alpha+\beta+\gamma+\delta}{4}=\frac{a_1}{a_0}$ $\ldots(4)$

Now $z^4+6Hz^2+4Gz+(a_0^2I-3H^2)=0$ has for its roots

$$(a_0\alpha+a_1) \text{ etc. or } a_0\left(\alpha+\frac{a_1}{a_0}\right) \text{ etc.}$$

or
$$a_0\left(\alpha-\frac{\alpha+\beta+\gamma+\delta}{4}\right) \text{ etc.}$$

or
$$\frac{a_0}{4}(3\alpha-\beta-\gamma-\delta) \text{ etc.}$$

Ex. 1. *If $\alpha, \beta, \gamma, \delta$ be the roots of*

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0,$$

then write down the values of the following :—

(i) $\Sigma(3\alpha-\beta-\gamma-\delta)(3\beta-\gamma-\delta-\alpha)$,

(ii) $\Sigma(3\alpha-\beta-\gamma-\delta)(3\beta-\gamma-\delta-\alpha)(3\gamma-\delta-\alpha-\beta)$,

(iii) $(3\alpha-\beta-\gamma-\delta)(3\beta-\gamma-\delta-\alpha)(3\gamma-\delta-\alpha-\beta)$
$\times(3\delta-\alpha-\beta-\gamma)$.

Proceed exactly as in Ex. 1 § 9 and the required values are

(i) $\frac{96H}{a_0^2}$, (ii) $\frac{-256}{a_0^3}G$, (iii) $\frac{256}{a_0^4}(a_0^2I-3H^2)$.

Ex. 2. (a) *Find the condition that the 2nd and 4th terms of the biquadratic* $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ *be removed by the same transformation.*

Let the transformed equation when the roots are diminished by h be $A_0y^4+4A_1y^3+6A_2y^2+4A_3y+A_4=0$.

If 2nd and 4th terms are removed by the same transformation, then the value of h which makes A_1 vanish should make A_3 zero. $A_1=a_0h+a_1=0$; $\therefore\ h=-\frac{a_1}{a_0}$ and for this value of h, $A_3=\frac{G}{a_0^2}$. Hence $G=0$ is the required condition.

(b) *Find the condition that 2nd and 3rd terms of the cubic* $a_0x^3+3a_1x^2+3a_2x+a_3=0$ *are removed by the same transformation.* **(Pb. 55)**

$H=a_0a_2-a_1^2=0$.

Ex. 3. *Find the condition that the biquadratic*

$$l\,(x^2+2px+q)^2+m\,(x^2+2px+q)+n$$

may be capable of being put in the form

$$l\,[(x+p)^2+(q-p^2)]^2+m\,[(x+p)^2+(q-p^2)]+n.$$

If we put $x+p=z$, then the transformed equation will not contain 2nd and 4th terms *i.e.* of z^3 and z and hence both 2nd and fourth terms are removed by the same transformation and as such G of the given biquadratic must vanish.

Ex. 4. *Prove that J vanishes for the equation*

$$m\,(x-n)^4-n\,(x-m)^4=0.$$

On simplification the above equation reduces to the form. $x^4+0x^3-6mnx^2+4mn\,(m+n)\,x-mn\,(m^2+n^2+mn)=0.$

From § 8, $J=\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}$ and putting the values of a_0, a_1 etc., we get

$$J=(mn)^2\begin{vmatrix} 1 & 0 & -mn \\ 0 & -1 & m+n \\ -1 & m+n & -(m^2+n^2+mn) \end{vmatrix}.$$

Now add R_1 to R_3 and then expand etc. $=0$.

§ 11. To form an equation whose roots are any powers of the roots of a given equation.

Let $f(x)\equiv(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots.(x-\alpha_n)$.

If we are to find an equation whose roots are pth powers of the roots of the given equation, then we should change x into

ax, bx, cx, $dx\ldots$in the above identity where a, b, c, $d\ldots.$ are the p roots of the equation $x^p-1=0$.

Multiply both members of the p identities thus obtained and we shall have

$$\phi(x)=(x^p-\alpha_1{}^p)(x^p-\alpha_2{}^p)(x^p-\alpha_3{}^p)\ldots(x^p-\alpha_n{}^p).$$

Now put $x^p=y$ in both sides and the equation thus obtained in y will have for its roots $\alpha_1{}^p$, $\alpha_2{}^p\ldots\alpha_n{}^p$.

Ex. 1. (*a*) *Find the equation whose roots are the squares of the roots of the equation* $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0$.

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=(x-\alpha_1)(x-\alpha_2).\;.(x-\alpha_n)\ldots(1)$$

Now the roots of $x^2-1=0$ are 1, -1. Replace x by $1.x$ and $-1.x$ in both sides of (1) and we get

$$(x^n+p_2x^{n-2}+\ldots)+(p_1x^{n-1}+p_3x^{n-3}+\ldots)$$
$$=(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

and

$$(-1)^n[(x^n+p_1x^{n-2}+\ldots)-(p_1x^{n-1}+p_3x^{n-3}+\ldots)]$$
$$=(-1)^n[(x+\alpha_1)(x+\alpha_2)\ldots(x+\alpha_n)].$$

Multiplying both sides of the above identities, we get

$$(x^n+p_2x^{n-2}+p_4x^{n-4}\ldots)^2-(p_1x^{n-1}+p_3x^{n-3}\ldots)^2$$
$$=(x^2-\alpha_1^2)(x^2-\alpha_2^2)\ldots(x^2-\alpha_n^2)$$

or $$x^{2n}+(2p_2-p_1^2)x^{2n-2}+(p_2^2-2p_1p_3+2p_4)x^{2n-4}+\ldots$$
$$=(x^2-\alpha_1^2)(x^2-\alpha_2^2)\ldots(x^2-\alpha_n^2).$$

It is clear that the powers of x in L. H. S. will be multiples of 2 and hence x^2 may be replaced by y and the required equation is

$$y^n+(2p_2-p_1^2)y^{n-1}+(p_2^2-2p_1p_3+2p_4)y^{n-2}+\ldots$$
$$=(y-\alpha_1^2)(y-\alpha_2^2)\ldots(y-\alpha_n^2).$$

Hence $y^n+(2p_2-p_1^2)y^{n-1}+\ldots=0$ is ths required equation whose roots are $\alpha_1^2, \alpha_2^2\ldots\alpha_n^2$.

(b) *Find the equation whose roots are the squares of the roots of the equation* $x^4+px^3+qx^2+rx+s=0$. **(Pb. 43)**

In practice we may however follow the method given below. Keep those terms of x whose powers are multiple of two on one side and the rest on the other side.

$x^4+qx^2+s=-x(px^2+r)$. Square and put $x^2=y$.

$\therefore\ (y^2+qy+s)^2=y(py+r)^2$. Simplifying, we get

$$y^4-(p^2-2q)y^3+(q^2-2pr+2s)y^2-(r^2-2qs)y+s^2=0$$

is the required equation.

(c) *If $\alpha, \beta, \gamma, \delta$ are the roots of the equation*

$$x^4+px^3+qx^2+rx+s=0.$$

find the value of

(i) $\Sigma\alpha^2$, *(ii)* $\Sigma\alpha^2\beta^2$, **(Pb. 35)** *and (c)* $\Sigma\alpha^2\beta^2\gamma^2$.

First find the equation whose roots are the spuares of the roots of the given equation and let its roots be A, B, C, D.

Hence we have to find ΣA, ΣAB and ΣABC of the new equation which from part (b) are respectively

$$p^2-2q,\ q^2-2pr+2s \text{ and } r^2-2qs.$$

(d) *Find the equation whose roots are the squares of the roots of the quartic equation* $x^4+bx^2+cx+d=0$.

Hence or otherwise show that if $\alpha, \beta, \gamma, \delta$ *are four numbers such that* $\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}+\sqrt{\delta}=0$, *then*

$$64\alpha\beta\gamma\delta-[4\Sigma\ \alpha\beta-(\Sigma\ \alpha)^2]^2=0. \qquad \textbf{(Pb. 56)}$$

Putting $p=0$, $q=b$, and $r=c$, $s=d$ in part (b) etc., we get

$$y^4+2by^3+(b^2+2d)\,y^2-(c^2-2bd)\,y+d^2=0\ldots \quad \ldots(1)$$

If the roots of the given equation be $\sqrt{\alpha}$, $\sqrt{\beta}$. $\sqrt{\gamma}$), $\sqrt{\delta}$; then $\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}+\sqrt{\delta}=0$ as the term of x^3 is missing. The roots of the new equation are therefore $\alpha, \beta, \gamma\ \delta$, Putting the values in the given relation from (1), we get

$$64.d^2-[4.(b^2+2d)-4b^2]^2=64d^2-64d^2=0.$$

Hence proved.

(e) *Calculate* $\Sigma\ \alpha_1{}^2=\alpha_2{}^2$ *for the general equation*

$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0$ **(Delhi Hon's 48)**

Ans. $(p_2{}^2-2p_1p_3+2p_4)$. Refer part (a) and (c).

(f) *Find the equation whose roots are the squares of the roots of the equation* $x^3+px^2+qx+r=0$ *and hence find the value of* $\Sigma\ \alpha^2\beta^2$ *for the given cubic,* α, β, γ *being its roots.* **(Pb. 40)**

Ans. $y^3-(p^2-2q)\,y^2+(q^2-2pr)\,y-r^2=0$.

$\Sigma\ \alpha^2\beta^2=(q^2-2pr)$.

(g) *If* α, β, γ *are the roots of* $x^3-ax^2+bx-c=0$, *then prove that the area of the triangle whose sides are* α, β, γ *is*

$$\tfrac{1}{4}\,[a\,(4ab-a^3-8c)]^{1/2}.$$

If case the triangle be right-angled then prove that

$$a\,(4ab-a^3-8c)\,(a^2-2b)=8c^2.$$

We know that if s be the semi-perimeter of a triangle, then its area $=\sqrt{[s\,(s-\alpha)\,(s-\beta)\,(s-\gamma)]}$

$$=[s\,(s^3-s^2\,(\alpha+\beta+\gamma)+s\,(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma]^{1/3}.$$

Now $\qquad s=\dfrac{\alpha+\beta+\gamma}{2}=\dfrac{a}{2}$.

$$\therefore \quad \text{area}=\left[\frac{a}{2}\left(\overset{???}{\underbrace{\frac{a^3}{8}-\frac{a^2}{4}\cdot\frac{a}{\boxed{2}}}}+\frac{a}{2}.b-c\right)\right]^{1/2}$$

$$=\tfrac{1}{4}\left[a\ (4ab-a^3-8c)\right]^{1/2}.$$

Again if the triangle be right-angled then $\alpha^2+\beta^2=\gamma^2$

or $$\alpha^2+\beta^2+\gamma^2=2\gamma^2$$

or $$(\alpha+\beta+\gamma)^2-2\,(\alpha\beta+\beta\gamma+\gamma\alpha)=2\gamma^2$$

or $$(a^2-2b)=2\gamma^2 \quad \text{or} \quad \gamma^2=\frac{a^2-2b}{2}.$$

But γ is a root of the given equation. Therefore

$$\gamma\ (\gamma^2+b)=a\gamma^2+c.$$

Putting for γ^2, we get

$$\sqrt{\left(\frac{a^2-2b}{2}\right)}\left[\frac{a^2-2b}{2}+b\right]=a\ \frac{(a^2-2b)}{2}+c$$

or $$\sqrt{\left(\frac{a^2-2b}{2}\right)}\ \frac{a^2}{2}=\frac{a}{2}\ (a^2-2b)+c.$$

Squaring $$\frac{a^2-2b}{2}\cdot\frac{a^4}{4}=\frac{a^2}{4}(a^2-2b)^2+ac\ (a^2-2b)+c^2$$

or $$a\ \frac{(a^2-2b)}{8}\left[a^3-2a\ (a^2-2b)-8c\right]=c^2$$

or $$a\ (a^2-2b)\ (4ab-a^3-8c)=8c^2$$

(h) *Find the condition that the cubic equation*

$$x^3-px^2+qx-r=0$$

should have a pair of roots of the form $\alpha\pm i\alpha$ *and show how to determine the roots in that case.* **(Agra 33, 37, 42)**

The third root will evidently be real and let it be β; then sum of the squares of the roots,

$$\beta^2+(\alpha+i\alpha)^2+(\alpha-i\alpha)^2=p^2-2q$$

or $$\beta^2+2\alpha^2+2i^2\alpha^2+2i\alpha^2-2i\alpha^2=p^2-2q$$

or $$\beta^2=p^2-2q.$$

But β is a root of the given equation.

$$\therefore\ \beta\ (\beta^2+q)=p\beta^2+r$$

or $\sqrt{(p^2-2q)}\ (p^2-2q+q)=p\ (p^2-2q)+$ squaring.

or $$(p^2-2q)\,(p^2-q)^2=p^2\,(p^2-2q)^2+2pr\,(p^2-2q)+r^2$$
or $$(p^2-2q)\,(p^4-2qp^2+q^2+p^4+2qp^2-2pr)=r^2$$
or $$(p^2-2q)\,(q^2-2pr)-r^2=0$$
is the required condition.

Also having found β we can divide the equation by $x-\beta$ and the remainnig roots can be found for the quadratic.

Ex. 2. (a) *Find the equation whose roots are the cubes of the roots of the equation*

$$x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\ldots+p_{n-1}x+p_n.$$

[Agra 42, 51]

$$x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\ldots+p_{n-1}x+p_n$$
$$\equiv(x-\alpha_1)\,(x-\alpha_2)\,(x-\alpha_3)\ldots(x-\alpha_n)$$

or $$(p_n+p_{n-3}x^3+\ldots)+x\,(p_{n-1}+p_{n-4}x^3+\ldots)$$
$$+x^2\,(p_{n-2}+p_{n-5}x^3\ldots)=(x-\alpha_1)\,(x-\alpha_2)\,(x-\alpha_3)\ldots(x-\alpha_n)$$

or $$P+Qx+Rx^2=(x-\alpha_1)\,(x-\alpha_2)\,(x-\alpha_3)\ldots(x-\alpha_n)\ldots\ldots(1)$$

where P, Q, R stand for expressions which involve powers of x that are multiples of 3 only.

Now the roots of $x^3-1=0$ are 1, ω and ω^2 where $1+\omega+\omega^2=0$ and $\omega^3=1$. $\therefore$ $\omega^4=\omega^3.\omega=\omega$. Replacing x by $1.x$, $\omega.x$, and $\omega^2.x$ in the above identity, we get

$$P+Qx+Rx^2\equiv(x-\alpha_1)\,(x-\alpha_2)\,(x-\alpha_3)\ldots(x-\alpha_n)\ldots\ldots(2)$$

$$P+Q\omega x+R\omega^2x^2\equiv(\omega x-\alpha_1)\,(\omega x-\alpha_2)\,(\omega x-\alpha_3)\ldots$$
$$(\omega x-\alpha_n)\ldots(3)$$

$$P+Q\omega^2x+R\omega x^2\equiv(\omega^2x-\alpha_1)\,(\omega^2x-\alpha_2)\,(\omega^2x-\alpha_3)\ldots$$
$$(\omega^2x-\alpha_n)\ldots(4)$$

Multiplying both sides of the above three identities, we get $P^3+Q^3x^3+R^3x^3-3PQRx^3=(x^3-\alpha_1^3)\,(x^3-\alpha_2^3)\ldots(x^3-\alpha_n^3)$.

The L. H. S. of above contains powers of x which are multiples of 3 and hence replacing x^3 by y in both sides, we get the required equation.

Note. $(a+b+c)\,(a+b\omega+c\omega^2)\,(a+b\omega^2+c\omega)$

$$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=a^3+b^3+c^3-3abc.$$

(b) *Find the equation whose roots are the cubes of the roots of the equation* $x^3+px^2+qx+r=0$ *and if* α, β, γ *be the roots of the given equation, find the value of* $\Sigma\,\alpha^3$ *and* $\Sigma\,\alpha^3\beta^3$.

(Agra 40, 52, 57)

When the equation is of 3rd degree we may follow the method given below. Keep those terms of x whose powers are multiples of 3 on one side and the rest on the other side,

$$\therefore\quad x^3+r=-(px^2+qx).$$

Now put $x^3=y$ or $x=y^{1/3}$

$$\therefore\quad y+r=-(py^{2/3}+qy^{1/3})\ldots\quad\ldots\quad\ldots(1)$$

Cubing both sides, we get

$$(y+r)^3=-[p^3y^2+q^3y+3pq\,y^{1/3}y^{2/3}\,(py^{2/3}+qy^{1/3})].$$

Put $py^{2/3}+qy^{1/3}=-(y+r)$ from (1).

$$\therefore\quad y^3+3y^2r+3yr^2+r^3=-[p^3y^2+q^3y-3pqy\,(y+r)].$$

Simplifying, we get the required equation as

$$y^3+(p^3-3pq+3r)\,y^2+(q^3-3pqr+3r^2)\,y+r^3=0.$$

The roots of the equation are $\alpha^3, \beta^3, \gamma^3$.

$$\therefore\quad \Sigma\alpha^3=-(p^3-3pq+3r) \text{ and } \Sigma\alpha^3\beta^3=(q^3-3pqr+3r^2).$$

(c) If α, β, γ *be the roots of the equation* $x^3+px^2-1=0$, *find* $\Sigma\alpha^3\beta^3$. **(Pb. 33)**

Proceed as above or put $q=0$, and $r=-1$ in $\Sigma\alpha^3\beta^3$ of part (a). **Ans.** 3.

(d) Find the equation whose roots are the cubes of the roots of the equation $x^3+3x^2+2=0$. **(Pb. 41)**

Ans. $y^3+33y^2+12y+8=0$.

(e) Find the equations whose roots are the cubes of the roots of the equation *(i)* $x^4-2x^3-x^2+2x+1=0$.

(ii) $x^4-x^3+2x^2+3x+1=0$. **(Agra 42)**

(i) $x\,(x^3+2)-x^2=2x^3-1$. Put $x^3=y$.

or $$y^{1/3}(y+2)-y^{2/3}=2y-1 \dots \quad \dots \quad \dots (1)$$

Cubing both sides,

$$y(y+2)^3-y^2-3y^{1/3}(y+2)y^{2/3}\left[y^{1/3}(y+2)-y^{2/3}\right]=(2y-1)^3.$$

Putting $y^{1/3}(y+2)-y^{2/3}=2y-1$. From (1).

we get

$$y(y+2)^3-y^2-3y(y+2)(2y-1)=(2y-1)^3.$$

Simplifying, $y^4-8y^3+14y^2+8y+1=0$ is the required equation.

(ii) $y^4+14y^3+50y^2+6y+1=0$.

(*f*) *Show that the roots of the eq.* $x^3-\frac{1}{3}q^2x-\frac{2}{27}q^3-r^2=0$ *differ by a constant from the squares of the corresponding roots of* $x^3+qx+r=0$.

First find the equation whose roots are the squares of the roots of the equation $x^3+qx+r=0$ which will be $y^3+2qy^2+q^2y-r^2=0$. Now remove its 2nd term by diminishing its roots by h, so that $na_0h+a_1=0$,

i.e. $h=\dfrac{-2q}{3}$ and we shall get the same equation as given, whose roots are therefore α^2-h, β^2-h, γ^2-h.

(*g*) *Show that the cubes of the roots of the equation*

$$x^3+ax^2+bx+ab=0$$

are given by the equation

$$x^3+a^3x^2+b^3x+a^3b^3=0. \qquad \textbf{(Agra 44)}$$

(*h*) *If* α *is a root of the equation* $x^4+ax^3-6x^2-ax+1=0$ *then show that* $\dfrac{1+\alpha}{1-\alpha}$ *is also a root. Hence show that the other two roots are* $-\dfrac{1}{\alpha}$, $\dfrac{\alpha-1}{\alpha+1}$. **(Pb. 58)**

Since α is a root, $\therefore\ \alpha^4+a\alpha^3-6\alpha^2-a\alpha+1=0 \dots \quad \dots (1)$

Putting $x=\dfrac{1+\alpha}{1-\alpha}$, we get

$$[(1+\alpha)^4+(1-\alpha)^4]+a(1+\alpha)(1-\alpha)[(1+\alpha)^2-(1-\alpha)^2]$$
$$-6(1+\alpha)^2(1-\alpha)^2=0$$

On simplification it reduces to 1. Hence $\dfrac{1+\alpha}{1-\alpha}$ is also a root.

Again if we replace x by $-1/x$, we get the same equation and therefore if α is a root then $-1/\alpha$ is also a root and if $-\dfrac{1}{\alpha}$ is a root then $\dfrac{1+(-1/\alpha)}{1-(-1/\alpha)}$. *i. e.* $\dfrac{\alpha-1}{\alpha+1}$ is also a root.

Proved.

§ 12. To find the equation whose roots are the symmetric functions of the roots of a given equation.

Let the given equation be $f(x)=0$ and the transformed equation be in y where x and y are connected by the relation $\phi(x, y)=0$. From the relation we find the value of y in terms of x and substitute in $f(x)=0$ in order to get the transformed equation. In general we have to eliminate x between $f(x)=0$ and $\phi(x, y)=0$ for getting the new equation.

Ex. 1. *(a) If* α, β, γ *are the roots of the equation* $x^3+px^2+qx+r=0$, *find the equation whose roots are* $\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$, *and hence find the value of*

(*i*) $\Sigma(\beta+\gamma)(\gamma+\alpha)$

and (*ii*) $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$. **(Pb. 44, 49)**

Let $y=\beta+\gamma=\alpha+\beta+\gamma-\alpha=-p-\alpha$.

$$\therefore \quad \alpha=-(y+p).$$

But α is a root of the given equation and hence this value of α will satisfy the given equation and therefore the required equation is

$$-(y+p)^3+p(y+p)^2-q(y+p)+r=0$$

or $y^3+2py^2+(p^2+q)y+(pq-r)=0$ and let its roots $\beta+\gamma$ etc. be denoted by A, B, C.

$\therefore$ (i) $\Sigma AB=(p^2+q)$, and (ii) $ABC=r-pq$.

(b) Find for the same cubic the equation whose roots are

$$\beta\gamma+\frac{1}{\alpha},\ \gamma\alpha+\frac{1}{\beta},\ \alpha\beta+\frac{1}{\gamma}.$$ **(Agra 58)**

$$y=\beta\gamma+\frac{1}{\alpha}=\frac{\alpha\beta\gamma+1}{\alpha}=\frac{1-r}{\alpha}\quad\therefore\quad\alpha=\frac{1-r}{y}\text{ etc.}$$

Ans. $ry^3+q(1-r)y^2-p(1-r)y+(1-r)^3=0.$

(c) *Find for the same cubic the equation whose roots are*

$$\frac{\alpha}{\beta+\gamma},\ \frac{\beta}{\gamma+\alpha},\ \frac{\gamma}{\alpha+\beta}$$

$$y=\frac{\alpha}{\beta+\gamma}=\frac{\alpha}{\beta+\gamma+\alpha-\alpha}=\frac{\alpha}{-p-\alpha}\quad\text{or}\quad\alpha=-\frac{py}{1+y}\text{ etc.}$$

Put the value of α in the given equation and simplifying, we get the required equation as

$$(r-pq)y^3+(3r-2pq-p^3)y^2+(3r-pq)y+r=0.$$

(d) *Find for the same cubic the equation whose roots are*

$\alpha(\beta+\gamma),\ \beta(\gamma+\alpha),\ \gamma(\alpha+\beta).$

$$y=\alpha\beta+\alpha\gamma=\alpha\beta+\beta\gamma+\gamma\alpha-\beta\gamma=q-\beta\gamma.$$

or $$y=q-\frac{\alpha\beta\gamma}{\alpha}=q+\frac{r}{\alpha}\quad\therefore\quad\alpha=\frac{r}{y-q}\text{ etc.}$$

Ans. $y^3-2qy^2+(pr+q^2)y+(r^2-pqr)=0.$

(e) *Find for the same cubic the equation whose roots are*

$$\frac{\alpha}{\beta+\gamma-\alpha},\ \frac{\beta}{\gamma+\alpha-\beta},\ \frac{\gamma}{\alpha+\beta-\gamma}.$$ **(Agra 57)**

Here $\alpha=-\dfrac{py}{1+2y}$ etc.

Ans. $(p^3-4pq+8r)y^3+(p^3-4pq+12r)y^2+(6r-pq)y+r=0.$

(*f*) *Find for the same cubic the equation whose roots are* $\beta\gamma-\alpha^2,\ \gamma\alpha-\beta^2,\ \alpha\beta-\gamma^2$. **(Pb. 37, 39)**

Also find the value of

$$\Sigma(\beta\gamma-\alpha^2)(\gamma\alpha-\beta^2) \text{ and } (\beta\gamma-\alpha^2)(\gamma\alpha-\beta^2)(\alpha\beta-\gamma^2).$$

$$y=\beta\gamma-\alpha^2=(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta-\gamma\alpha-\alpha^2$$

$$=q-\alpha(\alpha+\beta+\gamma)=q+p\alpha\ ; \quad \therefore \quad \alpha=\frac{y-q}{p}.$$

Ans. $y^3-y^2(3q^2-p^2)+y(3q^2-p^2q)+(rp^3-q^3)=0.$

(i) $3q^2-p^2q$. (ii) q^3-rp^3.

(*g*) *Find for the same cubic the equation whose roots are*

$$\frac{\alpha^2-\beta\gamma}{\alpha},\ \frac{\beta^2-\gamma\alpha}{\beta},\ \frac{\gamma^2-\alpha\beta}{\gamma}.$$

Here $y=-\dfrac{q-p\alpha}{\alpha}$ or $\alpha=-\dfrac{q}{y+p}$ etc. as in part (*f*).

$$r(p+y)^3-q^2(p+y)^2+pq^2(p+y)-q^3=0. \quad \textbf{Ans.}$$

(*h*) *If* α,β,γ *are the roots of* $x^3-x-1=0$, *find the equation whose roots are* $\dfrac{1+\alpha}{1-\alpha},\ \dfrac{1+\beta}{1-\beta},\ \dfrac{1+\gamma}{1-\gamma}$. *Hence write down the value of* $\Sigma\left(\dfrac{1+\alpha}{1-\alpha}\right)$.

Let $y=\dfrac{1+\alpha}{1-\alpha}$; $\therefore$ $y(1-\alpha)=(1+\alpha)$ or $\alpha=\dfrac{y-1}{y+1}$.

Putting in the given equation and simplifying we get the required equation as $y^3+7y^2-y+1=0$.

$\therefore$ $\Sigma\dfrac{1+\alpha}{1-\alpha}=$sum of the roots of the above equation$=-7$.

Ex. 2. (*a*) *If* α,β,γ *be the roots of the equation* $x^3+qx+r=0$, *find the equation whose roots are*

(*i*) $\dfrac{\beta+\gamma}{\alpha^2},\ \dfrac{\gamma+\alpha}{\beta^2},\ \dfrac{\alpha+\beta}{\gamma^2}$. **(Pb. 40)**

(*ii*) $\dfrac{\beta}{\gamma}+\dfrac{\gamma}{\beta},\ \dfrac{\gamma}{\alpha}+\dfrac{\alpha}{\gamma},\ \dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$. **(Agra 41, 50)** *(iii)* $\beta^2+\beta\gamma+\gamma^2,\ \gamma^2+\gamma\alpha+\alpha^2,\ \alpha^2+\alpha\beta+\beta^2.$

(i) $y=\dfrac{\beta+\gamma}{\alpha^2}=\dfrac{\alpha+\beta+\gamma-\alpha}{\alpha^2}=\dfrac{0-\alpha}{\alpha^2}=-\dfrac{1}{\alpha}.$

$\therefore\quad \alpha=-\dfrac{1}{y}$ etc.

$$ry^3-qy^2-1=0.$$ **Ans.**

(ii) $y=\dfrac{\beta}{\gamma}+\dfrac{\gamma}{\beta}=\dfrac{\beta^2+\gamma^2}{\beta\gamma}=\dfrac{(\beta+\gamma)^2-2\beta\gamma}{\beta\gamma}=\dfrac{\alpha^2}{\beta\gamma}-2.$

$$\because\quad \alpha+\beta+\gamma=0$$

or $$y+2=\frac{\alpha^3}{\alpha\beta\gamma}=\frac{\alpha^3}{-r}\ ;\quad \therefore\quad \alpha^3+r\,(y+2)=0.$$

Also $\alpha^3+q\alpha+r=0.$

Subtracting, we get $-q\alpha+r\,(y+1)=0$

or $$\alpha=\frac{r\,(y+1)}{q}\ \text{etc.}$$

Ans. $r^2\,(y+1)^3+q^3\,(y+1)+q^3=0.$

(iii) $y=\beta^2+\beta\gamma+\gamma^2=(\beta+\gamma)^2-\beta\gamma=\alpha^2-\beta\gamma$

$=\alpha^2+\gamma\alpha+\alpha\beta-(\alpha\beta+\beta\gamma+\gamma\alpha)=\alpha.\Sigma\alpha-\Sigma\alpha\beta$

$=0-q=-q.$ Similarly each root is q and hence the required equation is $(y+q)^3=0.$

(b) Prove that if $y=\alpha^2\,(\beta+\gamma)$ *where* α,β,γ *are the roots of* $x^3+qx+r=0$, *then* $y^3-3ry^2+(q^3+3r^2)\,y-r^3=0.$ **(Pb. 32)**

Ex. 3. *(a) If the roots of the equation*

$$x^3-6x^2+11x-6=0$$

be α,β,γ, *find the equation whose roots are*

$$\beta^2+\gamma^2,\ \gamma^2+\alpha^2,\ \alpha^2+\beta^2.$$ **(Pb. 44 45).**

First find the equation whose roots are the squares of the roots of the given equation and let its roots be A, B, C. Then from this equation find the equation whose roots are $B+C,\ C+A,\ A+B$ as in **Ex. 1 (a) P. 65.**

$$z^3-28z^2+245z-650=0.$$ **Ans.**

(b) If the roots of the equation $2x^3+x^2+x+1=0$ *be* α,β,γ, *find the equation whose roots are*

$$\frac{1}{\beta^2}+\frac{1}{\gamma^2}-\frac{1}{\alpha^2},\ \frac{1}{\gamma^2}+\frac{1}{\alpha^2}-\frac{1}{\beta^2},\ \frac{1}{\alpha^2}+\frac{1}{\beta^2}-\frac{1}{\gamma^2}.\ \checkmark$$

First find the equation whose roots are the squares of the roots of the given equation and in the new equation put $y=\frac{1}{u}$ and the equation thus obtained will have its roots $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$, $\frac{1}{\gamma^2}$ say A, B, C. Now we have to find an equation whose roots are $B+C-A$ etc.

$$z^3+z^2-13z+19=0. \qquad \textbf{Ans.}$$

(c) *If the roots of the cubic $x^3+2x^2+3x+1=0$ be α, β, γ, find the equation whose roots are*

$$\frac{1}{\beta^3}+\frac{1}{\gamma^3}-\frac{1}{\alpha^3},\ \frac{1}{\gamma^3}+\frac{1}{\alpha^3}-\frac{1}{\beta^3}\ \textit{and}\ \frac{1}{\alpha^3}+\frac{1}{\beta^3}-\frac{1}{\gamma^3}.$$

First find the equation whose roots are the cubes of the roots of the given equation

$$z^3+12z^2+172z-2072=0. \qquad \textbf{Ans.}$$

(d) *If the roots of the equation $x^3+px^2+qx+r=0$ be α, β, γ, find the equation whose roots are*

(i) $\beta^2+\gamma^2-\alpha^2$, $\gamma^2+\alpha^2-\beta^2$, $\alpha^2+\beta^2-\gamma^2$.

(ii) $\beta^2+\gamma^2$, $\gamma^2+\alpha^2$, $\alpha^2+\beta^2$.

Proceed as in parts (a) and (b). Students should be very careful in simplification which is quite lengthy and tedious.

(i) $$z^3-(p^2-2q)z^2-(p^4-4p^2q+8pr)z+p^6-6p^4q+8p^3r$$
$$+8p^2q^2-16pqr+8r^2=0.$$

(ii) $$z^3-2(p^2-2q)z^2+(p^4-4p^2q+5q^2-2pr)z$$
$$-(p^2q^2-2p^3r+4pqr-2q^3-r^2)=0.$$

Ex. 4. (a) *Find the equation whose roots are the ratios of the roots α, β, γ of the cubic $x^3+qx+r=0$.*

(Agra 21, 29, 41)

Let $y=\frac{\alpha}{\beta}$ (there will be six such ratios).

$\therefore\ \alpha=y\beta$. Now $f(\beta)=0$ and $f(\alpha)=0$. $\therefore\ f(\beta y)=0$

$$f(\beta)=\beta^3+q\beta+r=0,$$

$$f(\beta y)=\beta^3y^3+q\beta y+r=0.$$

Eliminating β between these two

$$\frac{\beta^3}{qr(1-y)}=\frac{\beta}{r(y^3-1)}=\frac{1}{qy(1-y^2)}.$$

$$\therefore\quad \beta=\frac{-r(1+y+y^2)}{qy(1+y)}\quad\text{and}\quad \beta^3=\frac{r}{y}\,\frac{1}{1+y}.$$

Now $\beta^3=(\beta)^3$.

$$\therefore\quad \frac{r}{y}\cdot\frac{1}{1+y}=-\frac{r^3(1+y+y^2)^3}{q^3.y^3(1+y)^3}$$

or

$$r^2(1+y+y^2)^3+q^3y^2(1+y)^2$$

is the required equation of sixth degree.

(b) If α, β, γ be the roots of $x^3+px^2+qx+r=0$, find the equation whose roots are $\alpha^2+2\beta\gamma$, $\beta^2+2\gamma\alpha$, $\gamma^2+2\alpha\beta$.

Let $y=\alpha^2+2\beta\gamma=\alpha^2+\dfrac{2\alpha\beta\gamma}{\alpha}=\alpha^2-\dfrac{2r}{\alpha}$.

$$\therefore\quad \alpha^3+0\alpha^2-\alpha y-2r=0 \quad \ldots\ \ldots\ \ldots(1)$$

Also

$$\alpha^3+p\alpha^2+q\alpha+r=0.. \quad \ldots\ \ldots\ \ldots(2)$$

Subtracting, we get

$$p\alpha^2+\alpha(q+y)+3r=0.. \quad \ldots\ \ldots\ \ldots(3)$$

Again multiplying (2) by 2 and adding to (1), we get

$$3\alpha^3+2p\alpha^2+(2q-y)\alpha=0$$

or

$$3\alpha^2+2p\alpha+(2q-y)=0. \quad \ldots\ \ldots\ \ldots(4)$$

Solving (3) and (4) by the method of cross-multiplication,

$$\frac{\alpha^2}{(q+y)(2q-y)-6pr}=\frac{\alpha}{qr-p(2q-y)}=\frac{1}{2p^2-3(q+y)}.$$

Eliminating α, we get

$$[qr-p(2q-y)]^2=[2q^2-6pr+qy-y^2]\,[2p^2-3q-3y],$$

Simplifying, we get

$$3y^3+y^2(-2p^2+3q-3q-p^2)+y(-6q^2+18pr+2p^2q-3q^2$$
$$-18pr+4p^2q)+(4p^2q^2-12p^3r-6q^3+18pqr$$
$$-81r^2-4p^2q^2+36pqr)=0$$

or $y^3-p^2y^2+q(2p^2-3q)y-(4p^3r-18pqr+2q^3+27r^2)=0.$

(c) If α, β, γ are the roots of $x^3+qx+r=0$, find the equation whose roots are $l\alpha+m\beta\gamma$, $l\beta+m\gamma\alpha$, $l\gamma+m\alpha\beta$.

Let $y=l\alpha+m\beta\gamma=l\alpha+m\dfrac{\alpha\beta\gamma}{\alpha}=l\alpha-\dfrac{mr}{\alpha}.$

$$\therefore\quad l\alpha^2-y\alpha-mr=0 \quad \ldots \quad \ldots \quad \ldots(1)$$

Multiplying both sides by α, we get

$$l\alpha^3-y\alpha^2-mr\alpha=0 \quad \ldots \quad \ldots \quad \ldots(2)$$

Also $\alpha^3+q\alpha+r=0$ as α is a root of the given equation.

$$\therefore\quad l\alpha^3+ql\alpha+rl=0. \quad \ldots \quad \ldots \quad \ldots(3)$$

Subtracting (2) and (3), we get

$$y\alpha^2+(ql+mr)\alpha+rl=0 \quad \ldots \quad \ldots \quad \ldots(4)$$

Solving (1) and (4) by cross multiplication, we get

$$\frac{\alpha^2}{-yrl+mr(ql+mr)}=\frac{\alpha}{-mry-rl^2}=\frac{1}{l(ql+mr)+y^2}.$$

Eliminating α, we get

$$r^2(my+l^2)^2=r(m^2r+lmq-yl)(ql^2+mrl+y^2)$$

or $y^3-mq\,y^2+(l^2q+3lmr)y+l^3r-l^2mq^2-2lm^2qr-m^3r^2=0.$

Ex. 5. *If α, β, γ be the roots of the cubic*

$$ax^3+3bx^2+3cx+d=0,$$

prove that the equation in y whose roots are

$$\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha},\quad \frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta},\quad \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}$$

is obtained by the homographic transformation

$$axy+b(x+y)+c=0. \qquad \textbf{(Pb. 45, 51)}$$

Also find the equation in y.

$$y=\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha}=\frac{\alpha\beta\gamma-\alpha^3}{\alpha(\alpha+\beta+\gamma-2\alpha)}=\frac{-\dfrac{d}{a}-\alpha^3}{\alpha\left(-\dfrac{3b}{a}-3\alpha\right)}$$

or
$$3yb\alpha+3ya\alpha^2=d+a\alpha^3$$
or
$$a\alpha^3-3ya\alpha^2-3yb\alpha+d=0.$$

Also $a\alpha^3+3b\alpha^2+3c\alpha+d=0$. Subtracting, we get
$$3\alpha^2(b+ay)+3\alpha(c+by)=0$$
or
$$x(b+ay)+(c+by)=0$$
i.e.
$$axy+b(x+y)+c=0.$$

Again $x=\dfrac{-(c+by)}{(b+ay)}$. Putting this value of x in the given equation and simplifying, we get the required equation.

***Ex. 6.** *If* α, β, γ *be the roots of the cubic*
$$a_0x^3+3a_1x^2+3a_2x+a_3=0,$$
find the equation whose roots are

$$(i)\quad \frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha},\ \frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta},\ \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}$$

(Agra 34, 35, 40, 46, 48, 53)

$$(ii)\quad \frac{2\beta\gamma-\alpha\beta-\alpha\gamma}{\beta+\gamma-2\alpha},\ \frac{2\gamma\alpha-\beta\gamma-\beta\alpha}{\gamma+\alpha-2\beta},\ \frac{2\alpha\beta-\gamma\alpha-\gamma\beta}{\alpha+\beta-2\gamma}.$$

(Agra 37, 45)

(i) We have already given one method in Ex. 5; alternative method is given below.

Diminish the roots of the given equation by h, and the transformed equation is $a_0y^3+3A_1y^2+3A_2y+A_3=0$ (§ 9 P. 52).

Its roots are $\alpha-h, \beta-h, \gamma-h$ *i.e.* say (a, b, c) Apply the condition that a, b, c are in G. P. and the condition obtained as in Ex. 4 (a) P. 24 is $A_1^3.A_3-a_0A_2^3=0$

or
$$(a_0h+a_1)^3(a_0h^3+3a_1h^2+3a_2h+a_3)$$
$$-a_0(a_0h^2+2a_1h+a_2)^3=0 \ldots \quad \ldots(2)$$

Above relation reduces to a cubic in h. But since $\alpha-h, \beta-h, \gamma-h$ are in G. P.,
$$\therefore\quad (\beta-h)^2=(\alpha-h)(\gamma-h)$$

or $$h=\frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta}=\text{required root.}$$

Hence replacing h by x in (2), we get the required cubic.

Simple Form. For the sake of simplicity let us put—

$$z=a_0x+a_1;$$

$$\therefore\quad a_0x^2+2a_1x+a_2=\frac{1}{a_0}\left[a_0^2x^2+2a_0a_1x+a_0a_2\right]$$

$$=\frac{1}{a_0}[(a_0x+a_1)^2+(a_0a_2-a_1^2)]=\frac{1}{a_0}[z^2+H].$$

Similarly $a_0x^3+3a_1x^2+3a_2x+a_3$

$$=\frac{1}{a_0^2}[z^3+3Hz+G]\quad[\text{see § 9, P. 53}],$$

and $$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4$$

$$=\frac{1}{a_0^3}[z^4+6Hz^2+4Gz+(a_0^2I-3H^2)].$$

Hence the required equation can be put in the form (see § 10 P. 55),

$$\frac{z^3}{a_0^2}[z^3+3Hz+G]-a_0\left(\frac{z^2+H}{a_0}\right)^3$$

or $$Gz^3-3H^2z^2-H^3=0,$$

where $z=a_0x+a_1$ and x *i.e.* h is the new root

(ii) Diminish the roots as above and apply the condition that the roots $\alpha-h$, $\beta-h$, $\gamma-h$ of the transformed equation be in H. P. and the condition obtained as in Ex. 5 (a) P. 25 is $a_0A_3^2-3A_1A_2A_3+2A_2^3=0$ which reduces to a cubic in h, whose value under the given condition of roots being in H. P. comes out to be any of the required roots. In the above condition replace h by x and putting $a_0x+a_1=z$, we get the required equation in simple form as

$$a_0\left(\frac{z^3+3Hz+G}{a_0^2}\right)^2-3z\left(\frac{z^2+H}{a_0}\right)\left(\frac{z^3+3Hz+G}{a_0^2}\right)$$
$$+2\left(\frac{z^2+H}{a_0}\right)^3=0$$

which reduces to the form

$$z^3-12H^2z^2-3HGz-(G^2+2H^3)=0$$

where $z=a_0x+a_1$ and x *i.e.* h is the new root.

Ex. 7. *The roots of the biquadratic*

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$$

are α, β, γ, δ *find the cubic whose roots are*

$$\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta},\ \frac{\gamma\alpha-\beta\delta}{\gamma+\alpha-\beta-\delta},\ \frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta}.$$

(Agra 46)

Diminish the roots by h and the transformed equation is $a_0y^4+4A_1y^3+6A_2y^2+4A_3y+A_4=0$ and its roots are $\alpha-h$, $\beta-h$, $\gamma-h$, $\delta-h$. Apply the condition that the product of two roots of this equation=product of other two as in Ex. 10 (a) P. 29.

$\therefore\ A_1^2A_4-a_0A_3^2=0$ which reduces to a cubic in h whose value under the given condition comes out to be any of the new roots. In the above condition replace h by x and a_0x+a_1 by z. We get the required equation in simple form as

$$z^2\frac{1}{a_0^3}\left[z^4+6Hz^2+4Gz+(a_0^2I-3H^2)\right]$$
$$-{}_0\left(\frac{z^3+3Hz+G}{a_0^2}\right)^2=0$$

or $$2Gz^3+(a_0^2I-12H^2)\,z^2-6HGz-G^2=0$$

where $z=a_0x+a_1$ and x *i.e.* h is the new root.

Ex. 8. (a) *If* α, β, γ *be the roots of the equation*

$$a_0x^3+3a_1x^2+3a_2x+a_3=0,$$

form the equation whose roots are

$$(\alpha-\beta)(\alpha-\gamma),\ (\beta-\gamma)(\beta-\alpha),\ (\gamma-\alpha)(\gamma-\beta).$$

(Agra 60)

Let us remove its 2nd term by diminishing the roots of the given equation by h *i.e.* $-\frac{a_1}{a_0}$ and the new equation is

$$y^3+\frac{3H}{a_0^2}y+\frac{G}{a_0^3}=0 \quad [\S\ 9,\ \text{P. } 53] \quad \ldots \quad \ldots(1)$$

Its roots are $\alpha-h$, $\beta-h$, $\gamma-h$ say A, B, C.

Now $\alpha-\beta=(\alpha-h)-(\beta-h)=A-B$.

Hence we have to find the equation whose roots are $(A-B)(A-C)$ etc. where A, B, C are roots of (1).

Let $z=(A-B)(A-C)=A^2-AB-AC+BC$,

or $z=A^2-\Sigma AB+2BC$

or $Az=A^3-\frac{A.3H}{a_0^2}-\frac{2G}{a_0^3}$. $\because$ $ABC=-\frac{G}{a_0^3}$.

Also $0=A^3+\frac{3H}{a_0^2}A+\frac{G}{a_0^3}$. $\because$ A is a root of (1).

Subtracting, we get $Az=-\frac{6HA}{a_0^2}-\frac{3G}{a_0^3}$

or $$A(a_0^2z+6H)=-\frac{3G}{a_0}$$

$$\therefore\ A=\frac{-3G}{a_0(a_0^2z+6H)}.$$

Putting this value of A in (1) and simplifying, we get the required equation as

$$z^3+\frac{9H}{a_0^2}z^2+\frac{27(G^2+4H^3)}{a_0^6}=0.$$

(*b*) *If* $x^3+3Hx+G=0$ *has its roots* α, β, γ, *then show that the roots of the equation* $x^3+9Hx^2-27(G^2+4H^3)=0$ *are*

$$(\alpha-\beta)(\alpha-\gamma),\ (\beta-\gamma)(\beta-\alpha),\ (\gamma-\alpha)(\gamma-\beta).$$

(Delhi Hons. 49)

(*c*) *If* (α, β, γ) *are the roots of* $x^3+3x+2=0$, *find the equation whose roots are as in part (a) or (b). Hence show that the cubic* $x^3+3x+2=0$ *has two imajinary roots.* **(Pb. 55)**

Proceeding as above $\alpha=-\frac{6}{z+6}$ and putting in the given equation, we get the required equation as

$$z^3+9z^2-216=0.$$

Product of all the roots $=-(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2=216$

or $\qquad (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2=-216 \quad$ *i.e.* $\;-$ive.

Therefore one of the three factors in L. H. S. must be $-$ive. If $(\alpha-\beta)^2=-$ive, then α and β should be conjugate complex. Hence two roots are imaginary.

Ex. 9. *If the roots of the equation*

$$a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{2!}a_2x^{n-2}+\ldots+a_n=0$$

be in arithmetical progression, show that they can be determined from the expression $-\frac{a_1}{a_0}\pm\frac{r}{a_0}\sqrt{\left(\frac{3(a_1^2-a_0a_2)}{(n+1)}\right)}$ *by giving r the values* $1, 3, 5\ldots(n-1)$ *when n is even and all the values* $0, 2, 4, 6\ldots(n-1)$ *when n is odd.* **(Agra 46, 60)**

Case I. If n be **odd,** then

When $n=3$, roots will be $\alpha-\delta, \alpha, \alpha+\delta$, *i.e.* $\alpha, \alpha\pm\delta$

or $$\alpha, \alpha\pm\frac{3-1}{2}\delta.$$

When $n=5$, roots will be $\alpha-2\delta, \alpha-\delta, \alpha, \alpha+\delta, \alpha+2\delta$

or $$\alpha, \alpha\pm\delta, \alpha\pm 2\delta$$

or $$\alpha, \alpha\pm\delta, \alpha\pm\frac{5-1}{2}\delta.$$

Hence the n roots of the equation when n is odd are

$$\alpha, \alpha\pm\delta, \alpha\pm 2\delta\ldots\alpha\pm\frac{n-1}{2}\delta.$$

Sum of the roots $=n\alpha=-\frac{na_1}{a_0}$, $\quad\therefore\quad \alpha=-\frac{a_1}{a_0}\ldots\quad\ldots(1)$

Sum of the squares of the roots $=p^2-2q$

or $\alpha^2+(\alpha\pm\delta)^2+(\alpha\pm 2\delta)^2+\ldots$

$$+\left(\alpha\pm\frac{n-1}{2}\delta\right)^2=\frac{n^2a_1^2}{a_0^2}-\frac{2n(n-1)}{2!}\frac{a_2}{a_0}$$

or $n\alpha^2+2\delta^2\left[1^2+2^2+3^2+\ldots+\left(\frac{n-1}{2}\right)^2\right]=\frac{n^2a_1^2}{a_0^2}-n(n-1)\frac{a_2}{a_0}$

Now $\Sigma n^2=\frac{n(n+1)(2n+1)}{6}$.

Put $n=\frac{n-1}{2}$ and $\alpha=-\frac{a_1}{a_0}$.

$$2\delta^2\cdot\frac{\frac{n-1}{2}\cdot\frac{n+1}{2}\cdot n}{6}=\frac{n^2a_1^2}{a_0^2}-n(n-1)\frac{a_2}{a_0}-n\cdot\frac{a_1^2}{a_0^2}$$

or $\delta^2\frac{(n-1)(n+1)n}{12}=n(n-1)\frac{a_1^2}{a_0^2}-n(n-1)\frac{a_2}{a_0}$.

or $\frac{\delta^2}{12}(n+1)=\frac{1}{a_0^2}(a_1^2-a_0a_2)$.

or $\delta^2=\frac{12(a_1^2-a_0a_2)}{a_0^2(n+1)}$ or $\delta=\frac{2}{a_0}\sqrt{\left[\frac{3(a_1^2-a_0a_2)}{(n+1)}\right]}$

Hence the roots are α, $\alpha\pm\delta$, $\alpha\pm 2\delta$, $\ldots\alpha\pm\frac{n-1}{2}\delta$.

or $\alpha\pm 0$, $\alpha\pm\frac{2}{a_0}\sqrt{\quad}$, $\alpha\pm 2.\frac{2}{a_0}\sqrt{\quad}$, $\alpha\pm 3.\frac{2}{a_0}\sqrt{\quad}$, $\ldots\alpha\pm\frac{n-1}{2}\cdot\frac{2}{a_0}\sqrt{\quad}$.

i.e. $-\frac{a_1}{a_0}\pm\frac{r}{a_0}\sqrt{\quad}$ where r takes the values 0, 2, 4, 6..$(n-1)$ when n is odd.

Case II. If n be **even,** then when $n=4$, roots will be $\alpha-3\delta$, $\alpha-\delta$, $\alpha+\delta$, $\alpha+3\delta$

or $\alpha\pm\delta$, $\alpha\pm 3\delta$ or $\alpha\pm\delta$, $\alpha\pm(4-1)\delta$.

Hence the roots of the equation when n is even are

$$\alpha\pm\delta,\ \alpha\pm 3\delta\ldots\alpha\pm(n-1)\delta.$$

Sum of the roots $=n\alpha=-\frac{na_1}{a_0}$; $\therefore\ \alpha=-\frac{a_1}{a_0}\ \ldots\ \ldots(1)$

Sum of the squares of the roots $=p^2-2q$

or $(\alpha\pm\delta)^2+(\alpha\pm3\delta)^2+\ldots[\alpha\pm(n-1)\delta]^2$

$$=\frac{n^2a_1^2}{a_0^2}-2\frac{n(n-1)}{2!}\cdot\frac{a_2}{a_0}$$

or $2\delta^2[\{1^2+2^2+3^2+4^2+\ldots(n-1)^2\}$

$$-\{2^2+4^2+6^2+\ldots+(n-2)^2\}]$$

$$=n(n-1)\frac{a_1^2}{a_0^2}-n(n-1)\frac{a_2}{a_0}.$$

Now $2^2+4^2+6^2+\ldots+(n-2)^2=2^2\left[1^2+2^2+3^2+\ldots+\left(\frac{n-2}{2}\right)^2\right].$

Putting $n=n-1$ and $\frac{n-2}{2}$ in the formula of Σn^2, we get

$$2\delta^2\left\{\frac{(n-1).n(2n-1)}{6}-2^2.\frac{\frac{n-2}{2}.\frac{n}{2}.(n-1)}{6}\right\}$$

$$=n(n-1)\frac{a_1^2}{a_0^2}-n(n-1)\frac{a_2}{a_0}$$

or $\delta^2=\frac{3}{a_0^2}\frac{(a_1^2-a_0a_2)}{n+1}$ $\therefore\ \delta=\pm\frac{1}{a_0}\sqrt{\left[\frac{3(a_1^2-a_0a_2)}{(n+1)}\right]}.$

Hence, roots are $\alpha\pm\frac{1}{a_0}\sqrt{\ }$, $\alpha\pm\frac{3}{a_0}\sqrt{\ }$, $\ldots\alpha\pm\frac{n-1}{a_0}\sqrt{\ }$

or $-\frac{a_1}{a_0}\pm\frac{r}{a_0}\sqrt{\ }$

where $r=1, 3, 5\ldots(n-1)$ when n is even.

§ 13. Equation of Squared Difference.

If α, β, γ be the roots of the equation

$$a_0x^3+3a_1x^2+3a_2x+a_3=0,$$

find the equation whose roots are the squares of the

differences of the roots of the given equation i. e. $(\alpha-\beta)^2$, $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$ and hence discuss the nature of the roots of the given equation.

$$a_0x^3+3a_1x^2+3a_2x+a_3=0\ \dots\ \dots\ \dots(1)$$

Its roots are α, β, γ.

Let us remove the 2nd term of (1) by diminishing its roots by h equal to $\frac{-a_1}{a_0}$ and the transformed equation is

$$y^3+\frac{3H}{a_0^2}y+\frac{G}{a_0^3}=0\dots\ \dots\ \dots\ \dots(2)$$

[§ 9 P. 53]

The roots of (2) are $\alpha-h$, $\beta-h$, $\gamma-h$, say A, B, C.

$\therefore\ (\alpha-\beta)^2=[(\alpha-h)-(\beta-h)]^2=(A-B)^2.$

Hence the equation of squared difference of (1) is the same as the equation of squared difference of (2).

Let the equation (2) be written as $x^3+qx+r=0\dots\ \dots(3)$ where $q=\frac{3H}{a_0^2}$ and $r=\frac{G}{a_0^3}$ whose roots are A, B and C and we are to find the equation whose roots are

$$(A-B)^2,\ (B-C)^2 \text{ and } (C-A)^2.$$

Let $z=(A-B)^2=(A+B)^2-4AB=(A+B)^2-4\frac{ABC}{C}$.

Now $A+B+C=0$; $\therefore\ A+B=-C$ and $ABC=-r$.

$$\therefore\ z=C^2+\frac{4r}{C} \quad\text{or}\quad C^3-Cz+4r=0.$$

But C is root of (3) ; $\therefore\ C^3+Cq+r=0$. Subtracting, we get $-C(z+q)+3r=0$ or $C=\frac{3r}{z+q}$.

Substituting the value of C in (3), we get

$$\frac{27r^3}{(z+q)^3}+q\cdot\frac{3r}{(z+q)}+r=0$$

or $$(z+q)^3+3q\,(z+q)^2+27r^2=0$$

or $$z^3+6qz^2+9q^2z+(27r^2+4q^3)=0 \quad \ldots \quad \ldots(4)$$

Putting the values of q and r in (4), we get the equation of squared difference of (2) and hence of (1) as

$$z^3+\frac{18H}{a_0^2}\,z^2+\frac{81H^2}{a_0^4}\,z+\frac{27}{a_0^6}\,(G^2+4H^3)=0\ldots \quad \ldots(5)$$

The roots of this equation are $(\alpha-\beta)^2$, $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$. If we multiply the roots of (5) by a_0^2, then the equation

$$z^3+18\,Hz^2+81H^2z+27\,(G^2+4H^3)=0 \ldots \quad \ldots(6)$$

will have for its roots $a_0^2\,(\alpha-\beta)^2$, $a_0^2\,(\beta-\gamma)^2$, $a_0^2\,(\gamma-\alpha)^2$.

Also from above we obtain that the product of the roots is $a_0^6\,(\beta-\gamma)^2\,(\gamma-\alpha)^2\,(\alpha-\beta)^2=-(27G^2+4H^3) \ldots \quad \ldots(7)$

$$=-27\Delta.$$

Note :—Result (4) and (7) should be committed to memory.

Discriminant : The function $\Delta=G^2+4H^3$ is called the discriminant of the cubic and its vanishing is the necessary and sufficient condition for the cubic to have two roots equal as will be seen in next article.

Nature of the roots.

Case I. $G^2+4H^3<0$ i. e. −ive. All roots real.

If G^2+4H^3 is −ive, then H should be −ive and $4H^3>G^2$. Now when H is −ive and G^2+4H^3 is −ive, then the terms in (5) or (6) are alternately +ive and −ive and hence the equation (5) must have all its roots +ive [see P. 21] *i. e.* $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$, $(\alpha-\beta)^2$ are all +ive which is possible only when α, β, γ all are real quantities.

Case II. $G^2+4H^3>0$ i. e. +ive. Two roots imaginary.

In this case product of all the roots *i. e.*

$$(\beta-\gamma)^2\,(\gamma-\alpha)^2\,(\alpha-\beta)^2=\frac{-27}{a_0^6}\,(G^2+4H^3) \text{ which is }-\text{ive.}$$

$$\therefore \quad (G^2+4H^3) > 0.$$

This is possible only if the equation has a −ive root say $(\alpha-\beta)^2=-$ive. ∴ $\alpha-\beta$ is pure imaginary. Hence α, β are conjugate complex. Hence two roots are imaginary.

Case III. $G^2+4H^3=0$. Two roots equal.

Under this condition product of all the roots is clearly zero and hence either of the roots $(\alpha-\beta)^2$, $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$ is zero *i. e.* two roots are equal

Case IV. $G=0$, $H=0$. All roots equal.

In this case the equation (5) reduces to $z^3=0$ *i.e.* all the three roots $(\alpha-\beta)^2$, $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$ vanish and hence $\alpha=\beta=\gamma$.

Ex. 1. *If α, β, γ be the roots of the equation*

$$a_0x^3+3a_1x^2+3a_2x+a_3=0,$$

find the condition that $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$, $(\alpha-\beta)^2$ be in (i) A. P. (ii) G. P. **(Agra 33, 44)**

Apply the condition that the roots A, B, C of the squared difference equation (5) be in A. P. or G. P.

(i) [Ex. 3 P. 23] $G^2+2H^3=0$.

(ii) [Ex. 4 P. 24] $8G^2+5H^3=0$.

Ex. 2. *If α, β, γ be the roots of the cubic*

$$x^3-3\,(1+a+a^2)\,x+(1+3a+3a^2+2a^3)=0,$$

prove that $(\beta-\gamma)\,(\gamma-\alpha)\,(\alpha-\beta)$ is a rational function of a.

From § 12, if $x^3+qx+r=0$ be the given equation, then the equation of squared difference is

$$y^3+6qy^2+9q^2y\,(27r^2+4q^3)=0.$$

$$\therefore \quad p^2=(\beta-\gamma)^2\,(\gamma-\alpha)^2\,(\alpha-\beta)^2=-(27r^2+4q^3)$$

where $\quad r=1+3a+3a^2+2a^3=(1+a+a^2)\,(1+2a)$

and $\quad q=-3\,(1+a+a^2)$.

$$\begin{aligned}\therefore \quad p^2&=-27\,(1+a+a^2)^2\,(1+2a)^2+27\times4\,(1+a+a^2)^3\\&=27\,(1+a+a^2)^2\,(4+4a+4a^2-1-4a-4a^2)\\&=81\,(1+a+a^2)^2.\end{aligned}$$

$$\therefore \quad p=(\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)=\pm9\,(1+a+a^2).$$

Ex. 3. *Form the equation of squared difference of* $x^3+6x^2+9x+4=0$. *What conclusions do you draw about the nature of the roots of the given equation?* **(Agra 47)**

Reduce the equation to the form $x^3+qx+r=0$ by diminishing its roots by -2 and then use § 13 result 4 and we get the required equation as $x^3-18x^2+81x=0$ whose one root is zero. ∴ $(\alpha-\beta)^2=0$ *i.e.* $\alpha=\beta$. Hence the two roots of the given equation are equal.

CHAPTER IV

SYMMETRIC FUNCTIONS OF THE ROOTS

§ 1. **Def. :** A function of the roots of a given equation is said to be symmetric if it remains unchanged when any two roots are interchanged. Below we shall find the values of some symmetric function for the cubic and the biquadratic. We shall try to express the given symmetric functions of the roots in terms of $\Sigma\alpha$, $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$, $\Sigma\alpha\beta\gamma\delta\ldots.$ whose values we can find from the given equation by § 1 P. 21.

Ex. 1. *Calculate the values of the following symmetric functions for the* **cubic** $x^3+px^2+qx+r=0$ *whose roots are* α, β, γ.

(Commit to memory the results of this example)

(i) $\Sigma\alpha^2\beta$. (ii) $\Sigma\alpha^2$. (iii) $\Sigma\alpha^3$. (iv) $\Sigma\alpha^2\beta\gamma$. (v) $\Sigma\alpha^2\beta^2$. (vi) $\Sigma\alpha^3\beta$.

$$\underset{6}{\Sigma}\alpha^2\beta=\alpha^2\beta+\alpha^2\gamma+\beta^2\gamma+\beta^2\alpha+\gamma^2\alpha+\gamma^2\beta$$

No. of terms 6.

Now $\Sigma\alpha.\Sigma\alpha\beta=(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$ No. of terms 9.

$$=\Sigma\alpha^2\beta+\underset{3\cdot 1}{3\alpha\beta\gamma}.$$

$$\therefore\quad \underset{6}{\Sigma}\alpha^2\beta=\underset{3\times 3}{\Sigma\alpha\,\Sigma\alpha\beta}-\underset{3\cdot 1}{3\alpha\beta\gamma}.$$ **(Remember)**

Both sides contain six terms.

$$=(-p)(q)-3(-r)=3r-pq.$$

The dight below every sigma denotes the number of terms in that symmetric function.

(ii) $\underset{3}{\Sigma}\alpha^2=\alpha^2+\beta^2+\gamma^2.$

Now $(\Sigma\alpha)^2=(\alpha+\beta+\gamma)^2=\underset{3}{\Sigma}\alpha^2+2\underset{2\times3}{\Sigma}\alpha\beta$ 9 terms.

$\therefore$ $\underset{(3)}{\Sigma}\alpha^2=\underset{9}{(\Sigma\alpha)^2}-2\underset{2\times3}{(\Sigma\alpha\beta)}=p^2-2q.$ **(Remember)**

Note. Above relation is true for equations of all degrees and will be used frequently

(iii) $\underset{3}{\Sigma}\alpha^3=\alpha^3+\beta^3+\gamma^3.$

Now $\underset{3}{\Sigma}\alpha.\underset{3}{\Sigma}\alpha^2=\underset{3}{(\alpha+\beta+\gamma)}\ \underset{3}{(\alpha^2+\beta^2+\gamma^2)}$

$$=\underset{3}{\Sigma}\alpha^3+\underset{6}{\Sigma}\alpha^2\beta.$$

$\therefore$ $\underset{3}{\Sigma}\alpha^3=\underset{3\times3}{\Sigma\alpha.\Sigma\alpha^2}-\underset{6}{\Sigma}\alpha^2\beta.$ **(Remember)**

The values of $\Sigma\alpha^2$ and $\Sigma\alpha^2\beta$ have already been calculated in (i) and (ii).

$$\therefore\quad \Sigma\alpha^3=(-p)(p^2-2q)-(3r-pq)$$
$$=3pq-p^3-3r.$$

(iv) $\underset{3}{\Sigma}\alpha^2\beta\gamma=\alpha^2\beta\gamma+\beta^2\gamma\alpha+\gamma^2\alpha\beta=\underset{1}{\alpha\beta\gamma}.\underset{3}{\Sigma}\alpha$

Hence $\underset{3}{\Sigma}\alpha^2\beta\gamma=\underset{1}{\alpha\beta\gamma}\underset{3}{\Sigma}\alpha=pr.$ **(Remember)**

(v) $\Sigma\alpha^2\beta^2=\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2.$ **(Pb. 55)**

Now $(\Sigma\alpha\beta)^2=(\alpha\beta+\beta\gamma+\gamma\alpha)^2=\underset{3}{\Sigma}\alpha^2\beta^2+2\underset{2\times3}{\Sigma}\alpha^2\beta\gamma$ 9 terms.

$\underset{3}{\Sigma}\alpha^2\beta^2=\underset{9}{(\Sigma\alpha\beta)^2}-2\underset{2\times3}{\Sigma}\alpha^2\beta\gamma.$ **(Remember)**

$=q^2-2pr$ from (iv).

(vi) $\underset{6}{\Sigma}\alpha^3\beta=\alpha^3\beta+\alpha^3\gamma+\beta^3\alpha+\beta^3\gamma+\gamma^3\alpha+\gamma^3\beta.$

Now $\underset{3}{\Sigma}\alpha^2 \underset{3}{\Sigma}\alpha\beta=(\alpha^2+\beta^2+\gamma^2)(\alpha\beta+\beta\gamma+\gamma\alpha)$

$$=\underset{6}{\Sigma}a^3\beta+\underset{3}{\Sigma}\alpha^2\beta\gamma$$

$\therefore \quad \Sigma\alpha^3\beta=\Sigma\alpha^2.\Sigma\alpha\beta-\alpha\beta\gamma.\Sigma\alpha.$ **(Remember)**

$$=q(p^2-2q)-pr=p^2q-2q^2-pr.$$

Ex. 2. *If α, β, γ be the roots of the equation*

$$a_0x^3+3a_1x^2+3a_2x+a_3=0,$$

find in terms of its coefficients the coefficients of the quadratic equation. **(Agra 41)**

$$(x-\alpha)^2(\beta-\gamma)^2+(x-\beta)^2(\gamma-\alpha)^2+(x-\gamma)^2(\alpha-\beta)^2=0.$$

The above equation when simplified takes the form

$$x^2\,\Sigma(\alpha-\beta)^2-2x.\Sigma\alpha(\beta-\gamma)^2+\Sigma\alpha^2(\beta-\gamma)^2=0$$

$$\Sigma(\alpha-\beta)^2=(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2=2\Sigma\alpha^2-2\Sigma\alpha\beta \text{ etc.}$$

$$\Sigma\alpha(\beta-\gamma)^2=\Sigma\alpha(\beta^2+\gamma^2-2\beta\gamma)=\Sigma\alpha^2\beta-6\alpha\beta\gamma \text{ etc.}$$

(Pb. 1938)

$$\Sigma\alpha^2(\beta-\gamma)^2=\Sigma\alpha^2(\beta^2+\gamma^2-2\beta\gamma)=2\Sigma\alpha^2\beta^2-2\Sigma\alpha^2\beta\gamma.$$

The values of all the symmetric functions have already been found in terms of $\Sigma\alpha$, $\Sigma\alpha\beta$ and $\alpha\beta\gamma$. Hence the equation is

$$(a_0a_2-a_1^2)x^2+(a_0a_3-a_1a_2)x+(a_1a_3-a_2^2)=0.$$

Ex. 3. *If α, β, γ be the roots of the cubic $x^3+px^2+qx+r=0$, find in terms of p, q, r the values of the following symmetric functions :—*

(i) $\dfrac{\beta^2+\gamma^2}{\beta\gamma}+\dfrac{\gamma^2+\alpha^2}{\gamma\alpha}+\dfrac{\alpha^2+\beta^2}{\alpha\beta}.$ **(Pb. 1953)**

(ii) $(\beta+\gamma-\alpha)^3+(\gamma+\alpha-\beta)^3+(\alpha+\beta-\gamma)^3$

(iii) $(\beta^3-\gamma^3)^2+(\gamma^3-\alpha^3)^2+(\alpha^3-\beta^3)^2.$

(iv) $\dfrac{\beta^2+\gamma^2}{\beta+\gamma}+\dfrac{\gamma^2+\alpha^2}{\gamma+\alpha}+\dfrac{\alpha^2+\beta^2}{\alpha+\beta}.$ **(Pb. 1936)**

(v) $\dfrac{\alpha^2+\beta\gamma}{\beta+\gamma}+\dfrac{\beta^2+\gamma\alpha}{\gamma+\alpha}+\dfrac{\gamma^2+\alpha\beta}{\alpha+\beta}$.

(vi) $\dfrac{2\beta\gamma-\alpha^2}{\beta+\gamma-\alpha}+\dfrac{2\gamma\alpha-\beta^2}{\gamma+\alpha-\beta}+\dfrac{2\alpha\beta-\gamma^2}{\alpha+\beta-\gamma}$.

(vii) $\alpha\,(\alpha+\beta)\,(\alpha+\gamma)+\beta\,(\beta+\gamma)\,(\beta+\alpha)+\gamma\,(\gamma+\alpha)\,(\gamma+\beta)$.

(viii) $\Sigma\left(\dfrac{\alpha-\beta}{\alpha+\beta}\right)^2$

(ix) $\left(\dfrac{1}{\alpha^2}-\dfrac{1}{\beta\gamma}\right)\left(\dfrac{1}{\beta^2}-\dfrac{1}{\gamma\alpha}\right)\left(\dfrac{1}{\gamma^2}-\dfrac{1}{\alpha\beta}\right)$. **(Pb. 1925)**

(i) The given symmetric function on simplification reduces to $\dfrac{\Sigma\alpha^2\beta}{\alpha\beta\gamma}$ etc. $=\dfrac{pq}{r}-3$ by Ex. 1.

(ii) Let $\beta+\gamma-\alpha=A$, $\gamma+\alpha-\beta=B$, $\alpha+\beta-\gamma=C$.

Also $B+C=2\alpha$, $C+A=2\beta$ and $A+B=2\gamma$ and $A+B+C=\alpha+\beta+\gamma$. We have to find the value of $A^3+B^3+C^3$.

$$A^3+B^3+C^3=(A+B+C)^3-3\,(A+B)\,(B+C)\,(C+A)$$
$$=(\Sigma\alpha)^3-3\,(2\alpha)\,(2\beta)\,(2\gamma)=24r-p^3.$$

(iii) Let us find the equation whose roots are α^3, β^3, γ^3 say A, B, C and by Ex. 2 (b) P. 63), the transformed equation is $y^3+(3r+p^3-3pq)\,y^2+(3r^2+q^3-3pqr)\,y+r^3=0$ and we have to find the value of $\Sigma\,(B-C)^2$.

$$=2\Sigma A^2-2\Sigma BC=2\,[(\Sigma A)^2-2\Sigma BC]-2\Sigma BC$$
$$=2\Sigma A^2-6\Sigma BC \text{ etc.}$$
$$=2p^6-12p^4q+12p^3r+18p^2q^2-18pqr-6q^3.$$

(4) The given symmetric function is

$$\frac{\Sigma\,(\beta^2+\gamma^2)\,(\alpha+\gamma)\,(\alpha+\beta)}{(\beta+\gamma)\,(\gamma+\alpha)\,(\alpha+\beta)}$$

$$N^r=\Sigma\,(\beta^2+\gamma^2)\,(\alpha^2+\Sigma\alpha\beta)=\Sigma\,(\beta^2+\gamma^2)\,(\alpha^2+q)$$
$$=2\Sigma\alpha^2\beta^2+2q\Sigma\alpha^2$$

whose value we have already found in Ex. 1.

$$\therefore\quad N^r=2p^2q-4pr-2q^2.$$

$D^r=(\beta+\gamma)\,(\gamma+\alpha)\,(\alpha+\beta)$. But $\alpha+\beta+\gamma=-p$. **(Pb. 55)**

$$\therefore\quad (\beta+\gamma)=-p-\alpha \text{ etc.}$$

$$\therefore\quad D^r=-(p+\alpha)\,(p+\beta)\,(p+\gamma)$$
$$=-[\,p^3+p^2\Sigma\alpha+p\Sigma\alpha\beta+\alpha\beta\gamma]=r-pq.$$ **(Pb. 38)**

(v) The given symmetric function is

$$=\frac{\Sigma\,(\alpha^2+\beta\gamma)\,(\alpha+\gamma)\,(\alpha+\beta)}{(\alpha+\beta)\,(\beta+\gamma)\,(\gamma+\alpha)}.$$

$$N^r=\Sigma\,(\alpha^2+\beta\gamma)\,(\alpha^2+q)=\Sigma\alpha^4+\Sigma\alpha^2\beta\gamma+q\Sigma\alpha^2+q\Sigma\alpha\beta,$$
$$\Sigma\alpha^4=(\Sigma\alpha^2)^2-2\Sigma\alpha^2\beta^2.$$

The values of all the symmetric functions involved have been calculated in Ex. 1, etc. and D^r we have already found in (iv).

$$\frac{N^r}{D^r}=\frac{p^4-3p^2q+5pr+q^2}{r-pq}.$$

(vi) The given symmetric function is

$$\frac{\Sigma\,(2\beta\gamma-\alpha^2)\,(\gamma+\alpha-\beta)\,(\alpha+\beta-\gamma)}{(\beta+\gamma-\alpha)\,(\gamma+\alpha-\beta)\,(\alpha+\beta-\gamma)}.$$

But $\qquad\alpha+\beta+\gamma=-p.$

$$\therefore\quad N^r=\Sigma\,(2\beta\gamma-\alpha^2)\,(p+2\beta)\,(p+2\gamma)$$
$$=\Sigma\,(2\beta\gamma-\alpha^2)\,[p^2+2p\,(\beta+\gamma)+4\beta\gamma]$$
$$=2p^2\Sigma\alpha\beta-p^2\Sigma\alpha^2+4p\,.\,\Sigma\alpha\beta\,(\alpha+\beta)-2p\Sigma\alpha^2\,(\beta+\gamma)$$
$$+8\Sigma\alpha^2\beta^2-4\Sigma\alpha^2\beta\gamma.$$

We have already calculated the values of all the symmetric functions in Ex. 1.

Also $\Sigma\alpha\beta\,(\alpha+\beta)=\Sigma\alpha^2\beta$ and $\Sigma\alpha^2\,(\beta+\gamma)=\Sigma\alpha^2\beta$.

$$D^r=(-p-2\alpha)\,(-p-2\beta)\,(-p-2\gamma)$$
$$=-(p^3+2p^2\Sigma\alpha+4p\Sigma\alpha\beta-8\alpha\beta\gamma)$$

$$\therefore\quad \frac{N^r}{D^r}=\frac{p^4-2p^2q+14pr-8q^2}{4pq-p^3-8r}.$$

(vii) The given function is

$$\Sigma\alpha\,(\alpha^2+\Sigma\alpha\beta)=\Sigma\alpha\,(\alpha^2+q)=\Sigma\alpha^3+q\Sigma\alpha$$

$$=2pq-p^3-3r \quad \text{[Ex. 1 (iii)].}$$

(viii) The given symmetric function is

$$\frac{(\beta\gamma-\alpha^2)\,(\gamma\alpha-\beta^2)\,(\alpha\beta-\gamma^2)}{\alpha^4\beta^4\gamma^4}$$

$$=\frac{(\Sigma\alpha\beta-\alpha.\Sigma\alpha)\,(\Sigma\alpha\beta-\beta.\Sigma\alpha)\,(\Sigma\alpha\beta-\gamma.\Sigma\alpha)}{\alpha^4.\beta^4.\gamma^4}$$

$$=\frac{1}{r^4}\,(q+p\alpha)\,(q+p\beta)\,(q+p\gamma)$$

$$=\frac{1}{r^4}\,(q^3+q^2p.\Sigma\alpha+qp^2.\Sigma\alpha\beta+p^3\alpha\beta\gamma)$$

$$=\frac{q^3-p^3r}{r^4}.$$

Ans.

(ix) $$\Sigma\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2=\Sigma\frac{(\alpha+\beta)^2-4\alpha\beta}{(\alpha+\beta)^2}=\Sigma\left\{1-4\frac{\alpha\beta}{(\alpha+\beta)^2}\right\}$$

$$=3-\frac{4}{\{(\alpha+\beta)\,(\beta+\gamma)\,(\gamma+\alpha)\}^2}\,\Sigma\{\alpha\beta\,(\gamma+\beta)^2\,(\gamma+\alpha)^2\}.$$

Now $\{(\gamma+\beta)^2\,(\gamma+\alpha)^2\}=(\gamma^2+\Sigma\alpha\beta)^2=(\gamma^2+q)^2$

$$=\gamma^4+2q\gamma^2+q^2.$$

$$\therefore\quad \Sigma\alpha\beta\,(\gamma+\beta)^3\,(\gamma+\alpha)^2=\Sigma\alpha\beta\,(\gamma^4+2q\gamma^2+q^2)$$

$$=\alpha\beta\gamma\Sigma\alpha^3+2q\alpha\beta\gamma\Sigma\alpha+q^2\Sigma\alpha\beta$$

$$=-r\,(3pq-p^3-3r)+2pqr+q^3 \qquad \text{[from Ex. 1, (iii)]}$$

$$=-pqr+p^3r+3r^2+q^3.$$

Also $(\alpha+\beta)^2\,(\beta+\gamma)^2\,(\gamma+\alpha)^2=(r-pq)^2$ [Ex. 3 (iv)]

$$\therefore\quad \Sigma\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2=3-\frac{4\,(q^3+3r^2+p^3r-pqr)}{(r-pq)^2}$$

$$=\frac{3p^2q^2-4p^3r-4q^3-2pqr-9r^2}{(r-pq)^2}$$

Ex. 4. *Express Σab and abc in terms of p, q and r where $p=\Sigma a$, $q=\Sigma a^2$, $r=\Sigma a^3$ and hence find the equation whose roots are (a, b, c).*

$$\Sigma a^2=(\Sigma a)^2-2\Sigma ab \quad \text{or} \quad q=p^2-2\Sigma ab; \therefore \ \Sigma ab=\frac{p^2-q}{2}$$

$$\Sigma a^3=\Sigma a.\Sigma a^2-\Sigma a^2b=p\ q-(\Sigma a.\Sigma ab-3abc)$$

or $$r=pq-p.\frac{p^2-q}{2}+3abc \quad \therefore \quad abc=\frac{p^3-3pq+2r}{6}.$$

Hence the equation whose roots are a, b, c, is

$$x^3-x^2.\Sigma a+x\ \Sigma ab-abc=0$$

or $$6x^3-6px^2+3\,(p^2-q)\,x+(3pq-p^3-2r)=0.$$

Ex. 5. (a) *Find the value of $\Sigma\ (a_0\alpha+a_1)^2\ (\beta-\gamma)^2$ in terms of the coefficients of the roots of the equation*

$$a_0x^3+3a_1x^2+3a_2x+a_3=0$$

whose roots are α, β, γ.

We know from § 9 P. 52 that the given equation reduces to the form $z^3+3Hz+G=0$ where $z=a_0x+a_1$. If the roots of (2) be denoted by z_1, z_2 and z_3, then $z_1=a_0\alpha+a_1$, $z_2=a_0\beta+a_1$ and $z_3=a_0\gamma+a_1$.

$\therefore$ $\dfrac{z_2-z_3}{a_0}=\beta-\gamma$ and hence we have to calculate the value of $\Sigma z_1^2.\dfrac{(z_2-z_3)^2}{a_0^2}$

$$=\frac{1}{a_0^2}\,[2\Sigma z_1^2z_2^2-2\Sigma z_1^2z_2z_3]$$

$$=\frac{1}{a_0^2}\,[2\{(\Sigma z_1z_2)^2-2\Sigma z_1^2z_2z_3\}-2\Sigma z_1^2z_2z_3]$$

$$=\frac{1}{a_0^2}\,[2.9H^2]=\frac{18}{a_0^2}\,(a_0a_2-a_1^2)^2$$

$$\because \quad \Sigma z^2_1z_2z_3=z_1z_2z_3\Sigma z_1=0 \quad \because \quad \Sigma z_1=0.$$

Ex. 6. *If $\alpha, \beta, \gamma, \delta$ be the roots of the equation*

$$x^4+px^3+qx^2+rx+s=0,$$

find the values of the following symmetric functions :—

(i) $\Sigma\,\alpha^2$. (ii) $\Sigma\,\alpha^2\beta$. (iii) $\Sigma\,\alpha^2\beta\gamma$. (iv) $\Sigma\,\alpha^2\beta^2$. (v) $\Sigma\,\alpha^3\beta$.

(vi) $\Sigma\,\alpha^4$. **(Pb. 35)** (vii) $\Sigma\,\dfrac{\alpha\beta}{\gamma^2}$. (viii) $\Sigma\,(\alpha-\beta)^4$.

(ix) $\Sigma\,(\alpha-\beta)^2$. **(Pb. 31)** (x) $\Sigma\,(\alpha-\beta)^2(\gamma-\delta)^2$. **(Pb. 52)**

(xi) $(\beta\gamma-\alpha\delta)(\gamma\alpha-\beta\delta)(\alpha\beta-\gamma\delta)$.

Ans. (i) p^2-2q as in Ex. 1.

(ii) $\underset{4\times 3}{\Sigma\,\alpha^2\beta}=\alpha^2(\beta+\gamma+\delta)+\ldots+\ldots+\ldots$

Now $\underset{4\times 6}{\Sigma\,\alpha.\Sigma\,\alpha\beta}=(\alpha+\beta+\gamma+\delta)(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$$=\underset{12}{\Sigma\,\alpha^2\beta}+3\underset{3\times 4}{\Sigma\,\alpha\beta\gamma}.$$

$\therefore\quad \Sigma\,\alpha^2\beta=\Sigma\,\alpha.\Sigma\,\alpha\beta-3\Sigma\,\alpha\beta\gamma$ **(Remember)**

$=3r-pq.$

(iii) $\underset{4\times 3}{\Sigma\,\alpha^2\beta\gamma}=\alpha^2(\beta\gamma+\beta\delta+\gamma\delta)+\ldots+\ldots+\ldots$

Now $\underset{4\times 3}{\Sigma\,\alpha}\;\Sigma\,\alpha\beta\gamma=\underset{4\times 4}{(\alpha+\beta+\gamma+\delta)}(\alpha\beta\gamma+\alpha\beta\delta+\gamma\delta\alpha+\gamma\delta\beta)$

$$=\underset{12}{\Sigma\,\alpha^2\beta\gamma}+\underset{4\times 1}{4\alpha\beta\gamma\delta}.$$

$\therefore\quad \Sigma\,\alpha^2\beta\gamma=\Sigma\,\alpha.\Sigma\,\alpha\beta\gamma-4\alpha\beta\gamma\delta.$ **(Remember)**

$=pr-4s.$

(iv) $\underset{6}{\Sigma\,\alpha^2\beta^2}=\alpha^2\beta^2+\alpha^2\gamma^2+\alpha^2\delta^2+\beta^2\gamma^2+\beta^2\delta^2+\gamma^2\delta^2.$

Now $\underset{36}{(\Sigma\,\alpha\beta)^2}=\underset{6\times 6}{(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)^2}$

$$=\underset{6}{\Sigma\,\alpha^2\beta^2}+2\underset{2\times 12}{\Sigma\,\alpha^2\beta\gamma}+\underset{6\times 1}{6\alpha\beta\gamma\delta}$$

$\therefore\quad \Sigma\,\alpha^2\beta^2=(\Sigma\,\alpha\beta)^2-2\Sigma\,\alpha^2\beta\gamma-6\alpha\beta\gamma\delta.$ **(Remember)**

$$=q^2-2(pr-4s)-6s \text{ from (iii)}=q^2-2pr+2s.$$

(v) $\underset{4\times 3}{\Sigma}\,\alpha^3\beta=\alpha^3(\beta+\gamma+\delta)+\ldots\ldots+\ldots\ldots+\ldots\ldots$

(Remember)

Now $\underset{4\times 6}{\Sigma\,\alpha^2}.\Sigma\,\alpha\beta=(\alpha^2+\beta^2+\gamma^2+\delta^2)$

$$\times(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)=\underset{12}{\Sigma}\,\alpha^3\beta+\underset{12}{\Sigma}\,\alpha^2\beta\gamma$$

$\therefore\quad \Sigma\,\alpha^3\beta=\Sigma\,\alpha^2.\Sigma\,\alpha\beta-\Sigma\,\alpha^2\beta\gamma.$ **(Remember)**

$=(p^2-2q)\,q-(pr-4s)$ from (ii) and (iii)

$=p^2q-2q^2-pr+4s$

(vi) $\Sigma\,\alpha^4=(\Sigma\,\alpha^2)^2-2\Sigma\,\alpha^2\beta^2$

$=(p^2-2q)^2-2(q^2-2pr+2s)$ from (i) and (iv)

$=p^4-4p^2q+2q^2+4pr-4s.$

(vii) In order to calculate $\Sigma\,\dfrac{\alpha\beta}{\gamma^2}$ we multiply

$$\Sigma\,\alpha\beta.\Sigma\,\frac{1}{\alpha^2}.$$

$$\Sigma\,\alpha\beta.\Sigma\,\frac{1}{\alpha^2}=(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)\times$$

$$\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}\right)$$

$$=\Sigma\,\frac{\alpha\beta}{\gamma^2}+\Sigma\,\frac{\alpha}{\beta}.$$

$$\therefore\quad \Sigma\,\frac{\alpha\beta}{\gamma^2}=\left(\Sigma\,\alpha\beta.\Sigma\,\frac{1}{\alpha^2}\right)-\Sigma\left(\frac{\alpha}{\beta}\right)\ \ldots\qquad\ldots(1)$$

Again for $\Sigma\,\dfrac{\alpha}{\beta}$ we multiply $\Sigma\,\alpha$ and $\Sigma\,\dfrac{1}{\alpha}$.

$$\Sigma\,\alpha.\Sigma\,\frac{1}{\alpha}=(\alpha+\beta+\gamma+\delta)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)$$

$$=\Sigma\,\frac{\alpha}{\beta}+4.$$

$$\therefore\quad \Sigma\,\frac{\alpha}{\beta}=\Sigma\,\alpha.\Sigma\,\frac{1}{\alpha}-4\ \ldots\qquad\ldots\qquad\ldots(2)$$

(Remember)

Now in order to find $\Sigma\frac{1}{\alpha}$ and $\Sigma\frac{1}{\alpha^2}$ we find the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ by putting $x=\frac{1}{x}$ in the given equation and the transformed equation is $sx^4+rx^3+qx^2+px+1=0$.

$$\therefore \quad \Sigma\frac{1}{\alpha}=\frac{-r}{s} \text{ and } \Sigma\left(\frac{1}{\alpha^2}\right)=\left(\Sigma\frac{1}{\alpha}\right)^2-2\Sigma\left(\frac{1}{\alpha\beta}\right)$$

$$=\frac{r^2}{s^2}-\frac{2q}{s}=\frac{r^2-2qs}{s^2}.$$

$$\therefore \quad \Sigma\frac{\alpha}{\beta}=-p\left(\frac{-r}{s}\right)-4=\frac{pr-4s}{s} \text{ and hence from (1).}$$

$$\frac{\Sigma\alpha\beta}{\gamma^2}=q\left(\frac{r^2-2qs}{s^2}\right)-\frac{pr-4s}{s}$$

$$=\frac{qr^2-2q^2s-prs+4s^2}{s^2}.$$

(viii) $\Sigma(\alpha-\beta)^4=3\Sigma\alpha^4-4\Sigma\alpha^3\beta+6\Sigma\alpha^2\beta^2.$

Put the values from (vi), (v), (iv) etc.

$$3p^4-16p^2q+20q^2+4pr-16s=0,.$$

(ix) $\Sigma(\alpha-\beta)^2=3\Sigma\alpha^2-2\Sigma\beta\gamma=3(p^2-2q)-2q=3p^2-8q.$

(x) $\Sigma(\alpha-\beta)^2(\gamma-\delta)^2=(\alpha-\beta)^2(\gamma-\delta)^2+(\gamma-\alpha)^2(\beta-\delta)^2$
$+(\beta-\gamma)^2(\alpha-\delta)^2$ **(Pb. 52)**

$$=(\alpha^2+\beta^2-2\alpha\beta)(\gamma^2+\delta^2-2\gamma\delta)+\ldots+\ldots$$
$$=2\Sigma\alpha^2\beta^2-2\Sigma\alpha^2\beta\gamma+12\alpha\beta\gamma\delta$$
$$=2(q^2-2pr+2s)-2(pr-4s)+12s$$
$$=2q^2-6pr+24s.$$

(xi) By actual multiplication we get the given function to be equal to $\Sigma\alpha^2\beta^2\gamma^2-\Sigma\alpha^2\beta\gamma\delta$

$$=(\Sigma\alpha\beta\gamma)^2-2\alpha\beta\gamma\delta\,.\,\Sigma\alpha\beta-\alpha\beta\gamma\delta\Sigma\alpha^2$$
$$=r^2-2qs-s(p^2-2q)=r^2-p^2s$$

Ex. 7. (a) *If* $\alpha, \beta, \gamma, \delta$ *be the roots of the equation* $x^4+px^3+qx^2+rx+s=0$, *find the equation whose roots are* $\beta\gamma+\alpha\delta$, $\gamma\alpha+\beta\delta$, $\alpha\beta+\gamma\delta$ *and hence find the value of*

$$(\alpha+\beta)\ (\alpha+\gamma)\ (\alpha+\delta)\ (\beta+\gamma)\ (\beta+\delta)\ (\gamma+\delta).$$

Let the roots be denoted by A, B, C.

$\therefore \quad \Sigma A=\Sigma\alpha\beta=q.$

$\Sigma AB=\underset{2\times 2}{(\beta\gamma+\alpha\delta)\ (\gamma\alpha+\beta\delta)}+\ldots+\ldots$ *i. e.* 12 terms

$=\underset{12}{\Sigma}\alpha^2\beta\gamma=pr-4s.$ [Ex. 6. (iii)] **(Pb. 1928)**

$ABC=$ By actual multiplication as in part (xi) Ex. 6

$=\Sigma\alpha^2\beta^2\gamma^2+\Sigma\alpha^3\beta\gamma\delta$ etc.

$=(r^2-2qs)+s(p^2-2q)=r^2-4qs+p^2s.$

Hence the equation is $x^3-(\Sigma A)x^2+(\Sigma AB)x-ABC=0$

or $\quad x^3-qx^2+(pr-4s)x-(p^2s-4qs+r^2)=0\ldots \qquad \ldots(1)$

Taking the product of $(\alpha+\beta)$ with the factor which contains other roots, we get

$$(\alpha+\beta)\ (\gamma+\delta)=(\gamma\alpha+\beta\delta)+(\beta\gamma+\alpha\delta)=A+B.$$

Thus we have to find the value of $(A+B)\ (B+C)\ (C+A)$.

$A+B=(A+B+C-C)=q-C$ from (1).

Hence we have to find the value of $(q-A)\ (q-B)\ (q-C)$

$=q^3-q^2\Sigma A+q\Sigma AB-ABC$

$=q^3-q^2.q+q(pr-4s)-(r^2-4qs+p^2s)$

$=pqr-r^2-p^2s.$ **Ans.**

(*b*) *If* $\alpha, \beta, \gamma, \delta$ *are the roots of the equation* $x^4+px^3+qx^2+rx+s=0$, *find the relation among the coefficients when the roots are so related that* $\alpha-\delta, \beta-\delta, \gamma-\delta$ *are in Harmonical Progression.*

According to the given condition,

$$\frac{1}{\beta-\delta}-\frac{1}{\alpha-\delta}=\frac{1}{\gamma-\delta}-\frac{1}{\beta-\delta},$$

or

$$\frac{\alpha-\beta}{\alpha-\delta}=\frac{\beta-\gamma}{\gamma-\delta}$$

or $$\alpha\gamma-\alpha\delta-\beta\gamma+\beta\delta=\alpha\beta-\alpha\gamma-\beta\delta+\gamma\delta$$

or $$2\gamma\alpha+2\beta\delta=\alpha\delta+\beta\gamma+\alpha\beta+\gamma\delta$$

or in terms of the notation of part (a), $2B=A+C$ where $A=\beta\gamma+\alpha\delta$ etc. and A, B, C are the roots of

$$x^3-qx^2+(pr-4s)x-(p^2s-4qs+r^2)-0 \text{ [by part (a)]}$$

Since $2B=A+C$ $\therefore$ $3B=A+B+C=q$; $\therefore$ $B=q/3$.

Hence the required condition is

$$\left(\frac{q}{3}\right)^3-q\left(\frac{q}{3}\right)^2+(pr-4s)\frac{q}{3}-(p^2s-4qs+r^2)=0$$

or $$9pqr-72qs-27p^2s-27r^2-2q^3=0.$$

Ex. 8. *If $\alpha, \beta, \gamma, \delta$ are the roots of $u\equiv x^4+qx^2+rx+s=0$, show that the equation whose roots are $\beta\gamma+\alpha\delta$, $\gamma\alpha+\beta\delta$, $\alpha\beta+\gamma\delta$ is $x^3-qx^2-4sx+4qs-r^2=0$ and that if x_1 is a root of this equation, then*

$$u\equiv\left(x^2-\frac{rx}{\lambda_1-\lambda_2}+\lambda_1\right)\left(x^2+\frac{rx}{\lambda_1-\lambda_2}+\lambda_2\right)$$

where λ_1 and λ_2 are the roots of $\lambda^2-\lambda x_1+s=0$. **(Agra 39)**

The first part follows by putting $p=0$ in the result of Ex. 7(a).

$$u\equiv x^4+qx^2+rx+s=0.$$

It does not contain the term of x^3 and hence sum of the four roots is zero. If sum of two is b, then sum of the other two is $-b$.

$$u\equiv(x^2+bx+\lambda_1)\,(x^2-bx+\lambda_2) \quad \ldots \quad \ldots (1)$$

Comparing the coefficients of like powers of x, we get

$$\lambda_1+\lambda_2-b^2=q \quad \text{or} \quad \lambda_1+\lambda_2=b^2+q \quad \ldots \quad \ldots (2)$$

$$b\,(\lambda_2-\lambda_1)=r \quad \text{or} \quad \lambda_2-\lambda_1=\frac{r}{b} \quad \ldots \quad \ldots (3)$$

$$\lambda_1\lambda_2=s \quad \text{or} \quad \lambda_1\lambda_2=s \quad \ldots \quad \ldots (4)$$

Now $$(\lambda_2-\lambda_1)^2=(\lambda_2+\lambda_1)^2-4\lambda_1\lambda_2$$

or $$\frac{r^2}{b^2}=(\lambda_2+\lambda_1)^2-4s$$

or $$\frac{r^2}{(\lambda_1+\lambda_2)-q}=(\lambda_2+\lambda_1)^2-4s \quad \text{from (2)}$$

or $$(\lambda_1+\lambda_2)^3-q\,(\lambda_1+\lambda_2)^2-4s\,(\lambda_1+\lambda_2)+4qs-r^2=0.$$

∴ $(\lambda_1+\lambda_2)$ is a root of $x^3-qx^2-4sx+4qs-r^2=0$.

But the root of the equation is given to be x_1.

$$\therefore \quad \lambda_1+\lambda_2=x_1 \qquad \ldots \quad \ldots 5)$$

Also $\lambda_1\lambda_2=s$. ∴ λ_1 and λ_2 are the roots of the equation $\lambda^2-\lambda x_1+s=0$. Also from (2), $b=\frac{r}{\lambda_2-\lambda_1}=-\frac{r}{\lambda_1-\lambda_2}$.

Substituting the value of b in (1), we get the required result.

Ex. 9. *If* $\alpha_1, \alpha_2, \ldots\ldots.\alpha_n$ *be the roots of the equation* $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0$, *find the values of the following symmetric functions :—*

(i) $\Sigma\,\frac{\alpha_1{}^2+\alpha_2{}^2}{\alpha_1\alpha_2}$.

(Agra 43, 58; Delhi Hon's 51, 54)

(ii) $\Sigma\,\frac{\alpha_1}{\alpha_2{}^2}$, (iii) $\Sigma\,(\alpha_1-\alpha_2)^2\alpha_3\alpha_4\ldots\alpha_n$.

(iv) $\Sigma\,\frac{\alpha_1{}^2}{\alpha_2}$. **(Agra 51)**

(i) $$\Sigma\,\frac{\alpha_1{}^2+\alpha_2{}^2}{\alpha_1\alpha_2}=\Sigma\left(\frac{\alpha_1}{\alpha_2}+\frac{\alpha_2}{\alpha_1}\right)$$

$$=\alpha_1\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\ldots+\frac{1}{\alpha_n}\right)-1$$

$$+\alpha_2\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\ldots+\frac{1}{\alpha_n}\right)-1$$

$$\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$+\alpha_n\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\ldots+\frac{1}{\alpha_n}\right)-1.$$

Adding, we get

$$\Sigma\alpha_1.\Sigma\frac{1}{\alpha}-n=(-p_1)\left(\frac{-p_{n-1}}{p_n}\right)-n=\frac{p_1p_{n-1}}{p_n}-n,$$

$\because$ $\frac{1}{\alpha_1}, \frac{1}{\alpha_2} \ldots \frac{1}{\alpha_n}$ are the roots of

$$p_nx^n+p_{n-1}x^{n-1}+\ldots+p_1x+1=0.$$

(ii) For calculating $\Sigma\frac{\alpha_1}{\alpha_2^2}$, we multiply $\Sigma\alpha_1$ and $\Sigma\frac{1}{\alpha_1^2}$.

$$\Sigma\alpha_1.\Sigma\frac{1}{\alpha_1^2}=(\alpha_1+\alpha_2+\ldots+\alpha_n)\left(\frac{1}{\alpha_1^2}+\frac{1}{\alpha_2^2}+\ldots+\frac{1}{\alpha_n^2}\right)$$

$$=\Sigma\frac{1}{\alpha_1}+\Sigma\frac{\alpha_1}{\alpha_2^2}.$$

$$\therefore\quad \Sigma\frac{\alpha_1}{\alpha_2^2}=\Sigma\alpha_1.\Sigma\frac{1}{\alpha_1^2}-\Sigma\frac{1}{\alpha_1}$$

$$=\Sigma\alpha_1\left[\left(\Sigma\frac{1}{\alpha_1}\right)^2-2\Sigma\frac{1}{\alpha_1\alpha_2}\right]-\Sigma\frac{1}{\alpha_1}.$$

$\frac{1}{\alpha_1}, \frac{1}{\alpha_2} \ldots \frac{1}{\alpha_n}$ are the roots of the equation

$$p_nx^n+p_{n-1}x^{n-1}+\ldots+p_1x+1=0.$$

$$\therefore\quad \Sigma\frac{\alpha_1}{\alpha_2^2}=-p_1\left(\frac{p^2_{n-1}}{p_n^2}-2\frac{p_{n-2}}{p_n}\right)+\frac{p_{n-1}}{p_n}$$

$$=\frac{p_{n-1}p_n-p_1p^2_{n-1}+2p_1p_{n-2}p_n}{p_n^2}.$$

(iii) $\Sigma(\alpha_1-\alpha_2)^2\alpha_3\alpha_4\ldots\alpha_n$

$$=\alpha_1\alpha_2\alpha_3\alpha_4\ldots\alpha_n\,\Sigma\frac{(\alpha_1-a_2)^2}{\alpha_1\alpha_2}$$

$$=(-1)^np_n.\left[\Sigma\left\{\frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2}-2\right\}\right].$$

Now we have n roots and we are taking two roots at a time *i. e.* there will be ${}^nC_2$ combinations and hence in the symmetric function 2 will occur ${}^nC_2$ times and the value of $\Sigma\,\dfrac{\alpha_1{}^2+\alpha_2{}^3}{\alpha_1\alpha_2}$ is already found in (i).

Hence $(-1)^n p_n\left[\dfrac{p_1p_{n-1}}{p_n}-n-2.\dfrac{n\,(n-1)}{2\,!}\right]$

$=(-1)^n\,(p_1p_{n-1}-n^2p_n).$ **Ans.**

$$\text{(iv)}\quad \Sigma\,\frac{\alpha_1{}^2}{\alpha_2}=\frac{1}{\alpha_1}\,(\alpha_1{}^2+\alpha_2{}^2+\ldots\alpha_n{}^2)-\alpha_1$$

$$+\frac{1}{\alpha_2}\,(\alpha_1{}^2+\alpha_2{}^2+\ldots\alpha_n{}^2)-\alpha_2$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$=\Sigma\alpha_1{}^2.\Sigma\frac{1}{\alpha_1}-\Sigma\alpha_1=(p_1{}^2-2p_2)\left(-\frac{p_{n-1}}{p_n}\right)+p_1$$

$$=p_1-\frac{p_{n-1}\,(p_1{}^2-2p_2)}{p_n}.$$

Ex. 10. (*a*) *Prove for the equation of Ex. 8, the following relation :—*

$$(\alpha_1{}^2+1)(\alpha_2{}^2+1)..(\alpha_n{}^2+1)=(1-p_2+p_4-..)^2+(p_1-p_3+\ldots)^2.$$

and hence prove that

$$(\alpha^2+1)(\beta^2+1)(\gamma^2+1)(\delta^2+1)=(1-q+s)^2+(p-r)^2$$

where $\alpha,\ \beta,\ \gamma,\ \delta$ *are the roots of the biquadratic of Ex. 7.*

(Agra 1941)

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n$$

$$\equiv(x-\alpha_1)\,(x-\alpha_2)\ldots(x-\alpha_n).$$

In the above identity put x equal to the roots of the equation $t^2+1=0$ *i. e.* $\pm i$. Putting $x=i$, we get

$$i^n+p_1i^{n-1}+p_2i^{n-2}+\ldots p_{n-1}.i+p_n$$

$$=(i-\alpha_1)\,(i-\alpha_2)\ldots(i-\alpha_n)$$

or $$i^n\left(1+\frac{p_1}{i}+\frac{p_2}{i^2}+\frac{p_3}{i^3}\ldots\right)$$

$$=(-1)^n\,(\alpha_1-i)\,(\alpha_2-i)\ldots(\alpha_n-i)$$

or $$i^n\,(1-p_1i-p_2+p_3i+p_4\ldots)$$

$$=(-1)^n\,(\alpha_1-i)\,(\alpha_2-i)\ldots(\alpha_n-i)$$

or
$$i^n \{(1-p_2+p_4\ldots)-i\,(p_1-p_3+p_5\ldots)\}$$
$$\equiv(-1)^n\,(\alpha_1-i)\,(\alpha_2-i)\ldots(\alpha_n-i).$$

Putting $x=-i$ and proceeding as above, we get
$$(-i)^n\,\{(1-p_2+p_4\ldots)+i\,(p_1-p_3+p_5\ldots)\}$$
$$\equiv(-1)^n\,(\alpha_1+i)\,(\alpha_2+i)\ldots(\alpha_n+i).$$

Multiplying both sides of the above identities, we get
$$(-i^2)^n\,\{(1-p_2+p_4\ldots)^2-i^2\,(p_1-p_3+p_5\ldots)^2\}$$
$$=(-1)^{2n}\,(\alpha_1^2+1)\,(\alpha_2^2+1)\ldots(\alpha_n^2+1)$$

or
$$(1-p_2+p_4\ldots)^2+(p_1-p_3+p_5\ldots)^2$$
$$=(\alpha_1^2+1)\,(\alpha_2^2+1)\ldots(\alpha_n^2+1).$$

For 2nd part proceed as above or find the equation whose roots are α^2+1, etc. by putting
$$z=x^2+1 \quad\text{or}\quad x=\sqrt{(z-1)}$$
and simplifying, we get the equation in z whose absolute term is our required value.

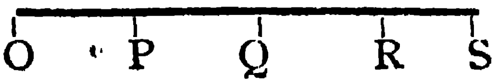
(b) If α, β, γ, δ be the roots of the equation
$$x^4-7x^3+8x^2-5x+10=0,$$
find the value of $(\alpha^2+2)\,(\beta^2+2)\,(\gamma^2+2)\,(\delta^2+2)$. **(Agra 1945)**

Proceed as above by any of the two methods.

Ex. 11. The distances on a right line of two pairs of points from a fixed origin on the line are the roots α, β and α', β' of the two quadratic equations
$$ax^2+2bx+c=0,\ a'x^2+2b'x+c'=0.$$

Prove that when one pair of points are the harmonic conjugates of the other pair, then the following relation exists :—
$$ac'+a'c-2bb'=0.$$

Let $OP=\alpha$, $OQ=\beta$ so that
$$\alpha+\beta=-\frac{2b}{a},\ \alpha\beta=\frac{c}{a}\ldots(1)$$



Again if $OR=\alpha'$, $OS=\beta'$; then

$$\alpha'+\beta'=-\frac{2b'}{a'},\quad \alpha'\beta'=\frac{c'}{a'}\quad \dots\quad \dots\quad \dots(2).$$

Since P, Q are the harmonic conjugates of R and S, then

$$\frac{1}{PR}+\frac{1}{PS}=\frac{2}{PQ}\qquad \text{[by definition]}$$

or
$$\frac{1}{\alpha'-\alpha}+\frac{1}{\beta'-\alpha}=\frac{2}{\beta-\alpha}$$

or
$$(\alpha'+\beta'-2\alpha)(\beta-\alpha)=2\{\alpha'\beta'-\alpha(\alpha'+\beta')+\alpha^2\}$$

or
$$(\alpha'+\beta')(\beta-\alpha)-2\alpha\beta+2\alpha^2=2\alpha'\beta'-2\alpha(\alpha'+\beta')+2\alpha^2$$

or
$$(\alpha'+\beta')(\beta-\alpha+2\alpha)=2\alpha\beta+2\alpha'\beta'$$

or
$$(\alpha'+\beta')(\alpha+\beta)=2(\alpha\beta+\alpha'\beta')$$

or
$$\left(-\frac{2b'}{a'}\right)\left(-\frac{2b}{a}\right)=2.\left(\frac{c}{a}+\frac{c'}{a'}\right).$$

or
$$2bb'=ac'+a'c.$$
Hence proved

§ 2. Newton's theorem on the sum of the powers of the roots of an equation. (Pb. 31, 36, 38, 42, 48)

The sums of similar powers of the roots of an equation can be expressed rationally in terms of the coefficients. (Agra 34, 35, 40, 41, 48, 49, 53, 57)

Let $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n$ be the roots of the equation

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_{n-1}x+a_n=0\quad \dots(1)$$

$$f(x)\equiv a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n).$$

Taking log of both sides and differentiating, we get

$$f'(x)=\frac{f(x)}{x-\alpha_1}+\frac{f(x)}{x-\alpha_2}+\dots+\frac{f(x)}{x-\alpha_n}.\quad \dots\quad \dots(2)$$

L. H. S. of (2) is

$$na_0x^{n-1}+(n-1)a_1x^{n-2}+(n-2)a_2x^{n-3}+\dots+a_{n-1}\quad (3).$$

In order to find out the R. H. S. we divide $f(x)$ by $(x-\alpha_1)$, $(x-\alpha_2)$, $(x-\alpha_3)\dots(x-\alpha_n)$ and add. We shall use here the process of synthetic division [§ 2·5 P. 6] and the remainders will be $f(\alpha_1), f(\alpha_2)\dots f(\alpha_n)$ each of which will be zero since $\alpha_1, \alpha_2 \ldots \alpha_n$ are the roots of $f(x)=0$.

$$\begin{array}{ccccc} a_0 & a_1 & a_2 & a_{n-1} & a_n \\ & a_0\alpha_1 & a_0\alpha_1^2+a_1\alpha_1 & & \\ \hline & a_0\alpha_1+a_1 & a_0\alpha_1^2+a_1\alpha_1+a_2 & a_0\alpha_1^{n-1}+a_1\alpha_1^{n-2}+\ldots+a_{n-1} & f(\alpha_1)=0 \end{array}$$

Hence the result of division of $f(x)$ by $(x-\alpha_1)$ gives the quotient

$$a_0x^{n-1}+(a_0\alpha_1+a_1)\,x^{n-2}+(a_0\alpha_1^2+a_1\alpha_1+a_2)\,x^{n-3}+\ldots$$
$$+(a_0\alpha_1^{n-1}+a_1\alpha_1^{n-2}+\ldots a_{n-1})=0$$

Similarly if we divide $f(x)$ by $(x-\alpha_2)\ldots(x-\alpha_n)$, we get

$$a_0x^{n-1}+(a_0\alpha_2+a_1)\,x^{n-2}+(a_0\alpha_2^2+a_1\alpha_2+a_2)\,x^{n-3}+\ldots$$
$$+(a_0\alpha_2^{n-1}+a_1\alpha_2^{n-2}+\ldots+a_{n-1})$$

... ...

... ...

... ...

$$a_0x^{n-1}+(a_0\alpha_n+a_1)\,x^{n-2}+(a_0\alpha_n^2+a_1\alpha_n+a_2)\,x^{n-3}+\ldots$$
$$+(a_0\alpha_n^{n-1}+a_1\alpha_n^{n-2}+\ldots a_{n-1}),$$

Adding all these quotients and denoting $\Sigma\alpha_1, \Sigma\alpha_1^2, \Sigma\alpha_1^3 \ldots \Sigma\alpha_1^n$ by $s_1, s_2, s_3 \ldots s_n$, we get the R. H. S. of (2) as

$$na_0x^{n-1}+(a_0s_1+na_1)x^{n-2}+(a_0s_2+a_1s_1+na_2)x^{n-3}+\ldots$$
$$+(a_0s_{n-1}+a_1s_{n-2}+\ldots na_{n-1}).$$

Equating the coefficients of like powers of x in (3) and (4), we get

$$a_0s_1+na_1=(n-1)a_1 \qquad \therefore \quad a_0s_1+a_1=0 \quad \ldots \quad \ldots(5)$$
$$a_0s_2+a_1s_3+na_2=(n-2)a_2 \qquad \therefore \quad a_0s_2+a_1s_1+2a_2=0 \ldots(6)$$
$$a_0s_3+a_1s_2+a_2s_1+na_3=(n-3)a_3$$
$$\therefore \quad a_0s_3+a_1s_2+a_2s_1+3a_3=0. \quad \ldots \quad \ldots \quad \ldots(7)$$
$$a_0s_{n-1}+a_1s_{n-2}+a_2s_{n-3}+\ldots+na_{n-1}=a_{n-1}$$
$$\therefore \quad a_0s_{n-1}+a_1s_{n-2}+a_2s_{n-3}+\ldots+(n-1)a_{n-1}=0 \quad \ldots(8)$$

From (5) we can find the value of s_1 and substituting in (6) the value of s_1 we can find the value of s_2. Again putting the values of s_1 and s_2 in (7), we can find the value of s_3 and continuing like this we can find the value of s_{n-1}. In this way we can find the value of $s_1, s_2 \dots s_{n-1}$ *i.e.* of s_r where r is an integer less than n. In case we are to find the value of s_r, where r is an integer greater than n, then we multiply $f(x)$ by x^{r-n} and we obtain

$$a_0x^r + a_1x^{r-1} + a_2x^{r-2} + \dots + a_{n-1}x^{r-n+1} + a_nx^{r-n} = 0.$$

Putting $x = \alpha_1, \alpha_2 \dots \alpha_n$ in above and adding, we get

$$a_0s_r + a_1s_{r-1} + a_2s_{r-2} + \dots + a_{n-1}s_{r-n+1} + a_ns_{r-n} = 0.$$

Now putting $r = n,\ n+1,\ n+2 \dots$, we get

$$a_0s_n + a_1s_{n-1} + a_2s_{n-2} + \dots + a_{n-1}s_1 + a_ns_0 = 0.$$

But $\qquad s_0 = 1 + 1 + 1 + 1 + \dots = n.$

$$\therefore \quad a_0s_n + a_1s_{n-1} + a_2s_{n-2} + \dots + a_{n-1}s_1 + na_n = 0 \dots\dots (9)$$

$$a_0s_{n+1} + a_1s_n + a_2s_{n-1} + \dots + a_{n-1}s_2 + a_ns_1 = 0 \quad \dots (10)$$

$$a_0s_{n+2} + a_1s_{n+1} + a_2s_n + \dots + a_{n-1}s_3 + a_ns_2 = 0 \quad \dots (11)$$

From the last relations we can find the values of $s_1, s_2 \dots s_{n-1}$ and putting in (9) we can get s_n and then from (10), we can get s_{n+1} and so on.

Again if we are to find the values of $\Sigma\alpha^{-1}$, $\Sigma\alpha^{-2}$, $\Sigma\alpha^{-3}$ *i. e.* of $s_{-1}, s_{-2}, s_{-3} \dots$, then we should find $s_1, s_2, s_3 \dots$ of the equation in which x has been replaced by $1/x$.

Note. Commit to memory results 5, 6, 7, 8, 9, 10, 11, and it may be noted that in all these relations the sum of the suffixes of a and s in each term is the same.

Ex. 1. *Find the value of $\Sigma\alpha^3$ for the equation*

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0. \qquad \textbf{(Agra 35)}$$

We have to find the value of s_3.

$$a_0s_1 + a_1 = 0 \qquad \therefore \quad s_1 = -p_1$$

$$a_0s_2 + a_1s_1 + 2a_2 = 0 \qquad \therefore \quad s_2 = p_1^2 - 2p_2$$

$$a_0s_3 + a_1s_2 + a_2s_1 + 3a_3 = 0$$

or $\qquad s_3 + p_1(p_1^2 - 2p_2) + p_2(-p_1) + 3p_3 = 0.$

$$\therefore \quad s_3 = 3p_1p_2 - p_1^3 - 3p_3.$$

Ex. 2. *α, β, γ are the roots of the equation $x^3 + px + q = 0$, show that $\Sigma\alpha^n = -q\Sigma\alpha^{n-3} - p\Sigma\alpha^{n-2}$.* **(Agra 38)**

$x^n + px^{n-1} + qx^{n-2} = 0$

Multiply the given equation by x^{n-3} and then put $x=\alpha$, β, γ and add, etc.

Ex. 3. *If* α, β, γ *be the roots of the equation* $x^3+3x+9=0$. *find the value of* s_9. **(Agra 38, 43, 47, 58)**

$$x^3=-3(x+3) \qquad \text{Cube both sides}$$

$$x^9=-27[x^3+9x^2+27x+27] \ldots \quad \ldots \quad \ldots (1)$$

Putting $x=\alpha$, β, γ and adding, we get

$$s_9=-27[s_3+9s_2+27s_1+27s_0], \; s_0=1+1+1=3.$$

s_1, s_2 and s_3 are easily calculated to be 0, -6, and -27.

Substituting the value of s_1, s_2 and s_3 and s_0 in above, we get $s_9=0$.

Ex. 4. (a) *If* α, β, γ *be the roots of the equation*

$$x^3+px^2+qx+r=0,$$

find the value of $\Sigma\alpha^3\beta^3$.

(Agra 34, 40, 52)

Let us find the equation whose roots are $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$.

$$z=\alpha\beta=\frac{\alpha\beta\gamma}{\gamma}=-\frac{r}{\gamma}.$$

$\therefore \quad \gamma=-\dfrac{r}{z}$. Put in the given equation and the transformed equation is $z^3-qz^2+prz-r^2=0$; we shall find s_3 for this equation, $s_1=q$, $s_2=(q^2-2pr)$.

Now $\qquad a_0s_3+a_1s_2+a_2s_1+3a_3=0,$

or $\qquad s_3-q(q^2-2pr)+pr.q-3r^2=0.$

$$\therefore \quad s_3=q^3-3pqr+3r^2.$$

(b) *Find the value of* s_4 *for the above cubic.* **(Pb. 49)**

$p^4-4p^2q+4pr+2q^2$. **Ans.**

Ex. 5. (a) *If* α, β, γ *be the roots of the equation* $x^3+2x+6=0$, *prove that* $s_7=2(s_4-s_6)$. **(Agra 45, 49)**

Clearly $s_1=0$. $s_2=-4$, and putting $x=\alpha$, β, γ and adding, we get $s_3+2s_1+6s_0=0$.

$$\therefore \quad s_3=-18.$$

Again $x^3=-2(x+3)$ ... ...(1)

Squaring, we get $x^6=4(x^2+6x+9)$. ... ...(2)

Putting $x=\alpha, \beta, \gamma$ and, adding, $s_6=4(s_2+6s_1+9s_0)=92.$

Multiplying both sides of (1) by x and putting $x=\alpha, \beta, \gamma$ and adding, $s_7=4(s_3+6s_2+9s_1)=-168.$

Multiplying both sides of (1) by x and putting $x=\alpha, \beta, \gamma$ and adding,

$$s_4=-2(s_2+3s_1)=8.$$

Now $2(s_4-s_6)=2(8-92)=-168=s_7$. Hence proved.

(b) *Prove that* $3s_2.s_5=5s_3.s_4$ *for the cubic* $x^3+qx+r=0$ *and find the value of* s_{-4}. **(Pb. 38)**

Also show that $\frac{s_5}{5}=\frac{s_3}{3}\cdot\frac{s_2}{2}$ **(Agra 45, Pb. 38)**

s_{-4} for the cubic $x^3+qx+r=0$ is s_4 for the cubic

$$\frac{1}{x^3}+\frac{q}{x}+r=0$$

i.e. for $rx^3+qx^2+1=0$ etc.

Find s_1, s_2, s_3 as usual and for s_4 multiply by x and putting $x=\alpha, \beta, \gamma$ and adding, we get,

$$\therefore \quad s_4=\frac{q^4+4r^2q}{r^4}.$$

(c) *If* $\alpha+\beta+\gamma=0$, *show that*

$$\alpha^5+\beta^5+\gamma^5=-5\alpha\beta\gamma(\alpha\beta+\beta\gamma+\gamma\alpha).$$
$$\alpha^7+\beta^7+\gamma^7=7\alpha\beta\gamma(\alpha\beta+\beta\gamma+\gamma\alpha)$$

Since $\alpha+\beta+\gamma=0$,

we can say that α, β, γ are the roots of $x^3+qx+r=0$

$$\therefore \quad \Sigma\alpha=0, \ \Sigma\alpha\beta=q, \ \alpha\beta\gamma=-r$$

and we have to prove that

$$S_5=5r.q, \quad \text{and} \quad S_7=-7rq \text{ etc. as in part (a).}$$

Ex. 6. *Find the value of* s_{-2} *and* s_5 *for the biquadratic* $x^4-x^3-19x^2+49x-30=0$. **(Pb. 36)**

Ans. $\frac{1261}{900}, -2849.$

Ex. 7. **(a)** *Calculate the value of s_8 for the equation $x^5-5px-1=0$.*

$$x^5=5px+1;$$

$$\therefore \quad x^4=(5p+x^{-1}).$$

Square and put $x=\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and add.

$$\therefore \quad s_8=25p^2.s_0+10ps_{-1}+s_{-2} \text{ etc.}$$

s_0 being equal to 5. **Ans.** $100p^2$

(b) Find the sum of the sixteenth powers of the roots of the equation $x^8+ax+b=0$.

Ans. $8b^2$.

(c) Show that sum of the twentieth powers of the roots of the equation $x^4+px+q=0$ is $50p^4q^2-4q^5$.

$$x^4=-(px+q); \quad \therefore \quad x^{20}=-(px+q)^5$$

$$=-(p^5x^5+5p^4qx^4+10p^3q^2x^3+10p^2q^3x^2+5pq^4x+q^5).$$

Putting $x=\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and adding, we get

$$s_{20}=-(p^5s_5+5p^4qs_4+10p^3q^2s_3+10p^2q^3s_2+5pq^4s_1+4q^5)$$

etc.

Ex. 8. **(a)** *If $\alpha+\beta+\gamma=6$, $\alpha^2+\beta^2+\gamma^2=14$, $\alpha^3+\beta^3+\gamma^3=36$, prove that $\alpha^4+\beta^4+\gamma^4=98$.* **(Agra 60)**

Let the equation whose roots are α, β, γ be

$$x^3+px^2+qx+r=0.$$

$$\therefore \quad -p=\alpha+\beta+\gamma=6, \quad \therefore \quad p=-6.$$

Also $\quad s_1=6, \ s_2=14, \ s_3=36.$

Again $\quad a_0s_2+a_1s_1+2a_2=0$

or $\quad 14+(-6)\,6+2q=0, \quad \therefore \quad q=11.$

$$a_0s_3+a_1s_2+a_2s_1+3a_3=0$$

or $\quad 36+(-6)\,14+11.6+3r=0, \quad \therefore \quad r=-6$

$$a_0s_4+a_1s_3+a_2s_2+a_3s_1=0$$

or $\quad s_4+(-6).36+11.14+(-6)\,6=0, \quad \therefore \quad s_4=98.$

(b) If $s_1=3$, $s_2=5$, $s_3=7$, show that $s_4=9$.

(c) If x, y, z are three numbers such that

$$x+y+z=1, \ x^2+y^2+z^2=2, \ x^3+y^3+z^3=3,$$

form the equation whose roots are x, y, z and hence or otherwise find the value of $x^5+y^5+z^5$. **(Pb 56).**

Ex. 9. *(a) Express $\Sigma\alpha\beta$ and $\alpha\beta\gamma$ in terms of a, b and c where $a=\alpha+\beta+\gamma$, $b=\alpha^2+\beta^2+\gamma^2$, $c=\alpha^3+\beta^3+\gamma^3$.*

If the equation whose roots are α, β, γ be taken to be $x^3+px^2+qx+r=0$, then you have to find the value of $\Sigma\alpha\beta$ *i. e* q and $\alpha\beta\gamma$ *i. e.* $-r$.

(b) If $\alpha+\beta+\gamma+\delta=0$, show that

$$\frac{\alpha^5+\beta^5+\gamma^5+\delta^5}{5}=\frac{\alpha^2+\beta^2+\gamma^2+\delta^2}{2}\cdot\frac{\alpha^3+\beta^3+\gamma^3+\delta^3}{3}.$$

Since $\alpha+\beta+\gamma+\delta=0$, let the equation be

$$x^4+qx^2+rx+s=0.$$

Now find s_5, s_2 and s_3 etc.

(c) Show that sum of the products of the first n integers taken three at a time is $n^2(n+1)^2(n-2)/48$.

The equation whose roots are the first n integers is

$$f(x)=(x-1)(x-2)(x-3)\ldots\ldots(x-n) \text{ (say)}$$
$$=x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\ldots=0$$

and we have to find the symmetric function of the type $\Sigma\alpha\beta\gamma=-p_3$ where $\alpha, \beta, \gamma\ldots$ means $1, 2, 3\ldots$respectively.

Now in usual notation,

$$s_1=1+2+3+4\ldots+n=\frac{n(n+1)}{2},$$

$$s_2=1^2+2^2+3^2+4^2+\ldots n^2=\frac{n(n+1)(2n+1)}{6},$$

$$s_3=1^3+2^3+3^3+4^3+\ldots+n^3=\left\{\frac{n(n+1)}{2}\right\}^2.$$

Now $s_1+p_1=0 \quad \therefore \quad p_1=-s_1=-\dfrac{n(n+1)}{2}$; $\quad\therefore\quad a_0=1.$

$$s_2+p_1s_1+2p_2=0 \quad\text{or}\quad \frac{n(n+1)(2n+1)}{6}-\frac{n^2(n+1)^2}{4}+2p_2=0$$

or $$p_2=\tfrac{1}{2}\left\{\frac{n^2(n+1)^2}{4}-\frac{n(n+1)(2n+1)}{6}\right\}.$$

$$s_3+p_1s_2+p_2s_1+3p_3=0$$

or $$\frac{n^2(n+1)^2}{4}-\frac{n(n+1)}{2}\,\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{4}$$
$$\times\left\{\frac{n^2(n+1)^2}{4}-\frac{n(n+1)(2n+1)}{6}\right\}+3p_3=0.$$

Combining 1st, 3rd and 2nd, 4th terms,

or $$\frac{n^2(n+1)^2}{4}\left\{1+\frac{n(n+1)}{4}\right\}-\frac{n^2(n+1)^2(2n+1)}{12}$$
$$\times(1+\tfrac{1}{2})+3p_3=0$$

or $$\frac{n^2(n+1)^2}{4}\left\{\frac{4-n^2-n}{4}-\frac{(2n+1)}{3}\cdot\frac{3}{2}\right\}+3p_3=0$$

or $$\frac{n^2(n+1)^2}{4}\left\{\frac{4+n^2+n-4n-2}{4}\right\}+3p_3=0.$$

$$\therefore\quad -p_3=\frac{n^2(n+1)^2}{48}(n^2-3n+2)$$

or $$\Sigma\alpha\beta\gamma=-p_3=\frac{n^2(n+1)^2(n-1)(n-2)}{48}.$$

Ex. 10. *For the equation*

$$f(x)=x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0,$$

find the value of $\Sigma(\alpha-\beta)^r$ *where* r *is an even number in terms of* $s_0, s_1, s_2\ldots$ **(Agra 50)**

If c_1, c_2, c_3 stand for binomial coefficients in the expansion of $(1+x)^r$, then

$$(\alpha-x)^r=\alpha^r-c_1\alpha^{r-1}.x+c_2\alpha^{r-2}.x^2\ldots-c_1\alpha x^{r-1}+x^r.$$

Putting $x=\alpha, \beta, \gamma\ldots k$ and adding, we get

$$(\alpha-\alpha)^r+(\alpha-\beta)^r+(\alpha-\gamma)^r+\ldots+(\alpha-k)^r$$
$$=s_0\alpha^r-c_1\alpha^{r-1}.s_1+c_2\alpha^{r-2}.s_2-\ldots-c_1\alpha.s_{r-1}+s_r\qquad\ldots(1)$$

$\because\quad \alpha^r+\alpha^r+\ldots n$ times$=n\alpha^r=s_0\alpha^r$.

Similarly write down the expansion of $(\beta-x)^r$ and proceeding as above, we get

$$(\beta-\alpha)^r+(\beta-\beta)^r+(\beta-\gamma)^r+\ldots+(\beta-k)^r$$
$$=s_0\beta^r-c_1\beta^{r-1}s_1+c_2\beta^{r-2}s_2-\ldots-c_1\beta s_{r-1}+s_r\ldots\qquad\ldots(2)$$

and so on we shall write n results corresponding to the n roots, and adding, we get

$$2\Sigma\,(\alpha-\beta)^r=s_0s_r-c_1s_{r-1}s_1+c_2s_{r-2}s_2-\ldots-c_1s_1s_{r-1}+s_0s_r$$
$$=2\,(s_0s_r-c_1s_{r-1}s_1+c_2s_{r-2}s_2-\ldots).$$
$$\therefore\quad \Sigma\,(\alpha-\beta)^r=s_0s_r-c_1s_{r-1}s_1+c_2s_{r-2}s_2-\ldots$$

Note :—Since r is even $(\alpha-\beta)^r=(\beta-\alpha)^r$ and hence while adding each term occurs twice, and hence we have written $2\Sigma\,(\alpha-\beta)^r$.

Ex. 11. *If $\alpha, \beta, \gamma\ldots$ are the roots of the equation*

$$x^n+\frac{x^{n-1}}{1\,!}+\frac{x^{n-2}}{2\,!}+\ldots+\frac{1}{n\,!}=0$$

show that $\Sigma\alpha^r=0$ for $r=2, 3, 4\ldots n$. **(Agra 48)**

Here we shall show that $\Sigma\alpha\neq0$ and $\Sigma\alpha^r$ for values of r greater than n also does not vanish.

$$a_0S_1+a_1=0;\quad\therefore\quad S_1=-1;$$
$$a_0S_2+a_1S_1+2a_2=0\quad\text{or}\quad S_2=-1+2\,.\frac{1}{2\,!}=0.$$
$$\therefore\quad S_2=0.$$

Similarly $S_3=S_4=S_5=\ldots S_n=0,$

Again $a_0S_{n+1}+a_1S_n+a_2S_{n-1}+\ldots a_nS_1=0.$

or
$$S_{n+1}+0+0+0+\ldots+\frac{1}{n\,!}(-1)=0.$$
$$\therefore\quad S_{n+1}=\frac{1}{n\,!}.$$

Similarly $S_{n+2}, S_{n+3}\ldots$ do not vanish.

Ex. 12. *The sum of rth powers of the roots of the equation $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0$ is denoted by s_r and $S_m=s_1+s_2+\ldots+s_m$. Prove that if S_m have a finite limit when $m\to\infty$, that limit is*

$$\frac{-(p_1+2p_2+\ldots+np_n)}{(1+p_1+p_2+\ldots+p_n)}\,.\qquad\textbf{(Agra)}$$

$$s_1+p_1=0$$
$$s_2+s_1p_1+2p_2=0$$

$$s_3+s_2p_1+s_1p_2+3p_3=0$$

$$s_n+s_{n-1}p_1+s_{n-2}p_2+\dots+np_n=0$$

$$s_{n+1}+s_np_1+s_{n-1}p_2+\dots+s_1p_n=0$$

$$s_m+s_{m-1}p_1+s_{m-2}p_2+\dots+s_{m-n}p_n=0.$$

Adding all these and putting $s_1+s_2+\dots s_m=S_m$, we get

$$(S_m+p_1S_{m-1}+p_2S_{m-2}+\dots p_nS_{m-n})+(p_1+2p_2+3p_3+\dots+np_n)=0.$$

Now let the limit of S_m as $m\to\infty$ be λ and hence limit of S_{m-1} etc. will also be λ.

$\therefore \quad \lambda\,(1+p_1+p_2+\dots+p_n)+(p_1+2p_2+3p_3+\dots+np_n)=0.$

$\therefore \quad \lambda$ is etc. etc.

Ex. 13. *If $\alpha, \beta, \gamma\dots$ be the roots of the equation $f(x)=0$ prove that $\dfrac{f'(x)}{f(x)}=\Sigma S_{m-1}x^{-m}$ where S_m is the sum of the mth powers of the roots.* **(Nagpur 57)**

$$f(x)=(x-\alpha)\,(x-\beta)\,(x-\gamma)\dots n \text{ factors.}$$

Taking log of both sides,

$$\log f(x)=\log(x-\alpha)+\log(x-\beta)+\log(x-\gamma)+\dots$$

Differentiating w. r. t. x, we get

$$\frac{1}{f(x)}\cdot f'(x)=\frac{1}{x-\alpha}+\frac{1}{x-\beta}+\frac{1}{x-\gamma}+\dots \qquad \dots\dots(1)$$

Now $\quad \dfrac{1}{x-\alpha}=\dfrac{1}{x}\left(1-\dfrac{\alpha}{x}\right)^{-1}=\dfrac{1}{x}\left(1+\dfrac{\alpha}{x}+\dfrac{\alpha^2}{x^2}+\dfrac{\alpha^3}{x^3}+\dots\right)$

$$=\frac{1}{x}+\frac{\alpha}{x^2}+\frac{\alpha^2}{x^3}+\frac{\alpha^3}{x^4}+\dots$$

$$\therefore \quad \frac{f'(x)}{f(x)}=\frac{1}{x}+\frac{\alpha}{x^2}+\frac{\alpha^2}{x^3}+\frac{\alpha^3}{x^4}+\dots$$

$$+\frac{1}{x}+\frac{\beta}{x^2}+\frac{\beta^2}{x^3}+\frac{\beta^3}{x^4}+\dots+\text{for } n \text{ roots}$$

$$=n\frac{1}{x}+\frac{S_1}{x^2}+\frac{S_2}{x^3}+\frac{S_3}{x^4}+\dots$$

Now $S_0=1+1+1+\ldots=n.$

$$\frac{f'(x)}{f(x)}=S_0x^{-1}+S_1x^{-2}+S_2x^{-3}+S_3x^{-4}+\ldots$$

$$=\Sigma S_{m-1}x^{-m}.$$

Proved

§ 3. Prove that every rational symmetric function of the roots of an algebraic equation can be expressed rationally in terms of the coefficients. (Agra)

If the given symmetric function be fractional then it can be reduced to a fraction, whose numerator and denominator both are integral symmetric functions. Every integral function of the roots of the equation will contain terms of the type $K\alpha^p\beta^q\gamma^r\ldots$ where p, q, r may be equal or some of them equal or all equal and the function can be written $K\Sigma\alpha^p\beta^q\gamma^r\ldots$ We already know by § 2 that $\Sigma\alpha^r$ can be expressed in terms of the coefficients. Here we shall prove that symmetric functions of the type $K\Sigma\alpha^p\beta^q\gamma^r\ldots$can also be expressed in terms of the coefficients.

We shall first express $\Sigma\alpha^p\beta^q$.

Let $s_p=\alpha^p+\beta^p+\gamma^p+\ldots$

and $s_q=\alpha^q+\beta^q+\gamma^q+\ldots$

$$\therefore \quad s_p.s_q=(\alpha^{p+q}+\beta^{p+q}+\gamma^{r+q}\ldots)+(\alpha^p.\beta^q+\beta^p.\alpha^q+\ldots)$$

or
$$s_p.s_q=s_{p+q}+\Sigma\alpha^p.\beta^q,$$

$$\therefore \quad \Sigma\alpha^p.\beta^q=s_p.s_q-s_{p+q} \quad \ldots \quad \ldots \quad \ldots(1)$$

(Remember)

The value of s_p and s_q and s_{p+q} can be found in terms of the coefficients by § 2.

Note. In case the exponents p and q become equal, then terms of the type $\alpha^p\beta^q$ and $\alpha^q.\beta^p$ become equal.

$$\therefore \quad 2\Sigma\alpha^p.\beta^p=(s_p)^2-s_{2p}.$$

or
$$\Sigma\alpha^p.\beta^p=\frac{1}{2\,!}\,[(s_p)^2-S_{2p}] \ldots \quad \ldots \quad \ldots(2)$$

(Remember)

Now we shall express $\Sigma\alpha^p.\beta^q.\gamma^r$.

$$\Sigma\alpha^p.\beta^q=\alpha^p\beta^q+\alpha^q.\beta^p+\alpha^p\gamma^q+\gamma^p\alpha^q+\ldots$$

and $$s_r=\alpha^r+\beta^r+\gamma^r+\ldots$$

Multiplying the two, we get

$$(\Sigma\alpha^p.\beta^q)\ s_r=\Sigma\alpha^{r+r}\beta^q+\Sigma\alpha^{q+r}\beta^p+\Sigma\alpha^p.\beta^q.\gamma^r.$$

Now use result (1) for the double function

$$(s_p.s_q-s_{p+q})\ s_r=[s_{p+r}.s_q-s_{p+q+r}]$$
$$+[s_{q+r}s_p-s_{p+q+r}]+\Sigma\alpha^p.\beta^q.\gamma^r.$$

$$\therefore\quad \Sigma\alpha^p.\beta^q.\gamma^r=s_p.s_q.s_r+2s_{p+q+r}$$
$$-s_{p+q}.s_r-s_{q+r}.s_p-s_{r+p}.s_q.$$ **(Remember)**

The value of s_p etc. can be calculated as in § 2 in terms of the coefficients. Hence the theorem is proved.

In case $p=q=r$, then the 3 ! terms obtained by interchanging the powers p, q, r in $\alpha^r\beta^1\gamma^r$ become equal and hence

$$\Sigma\alpha^p.\beta^p.\gamma^p=\frac{1}{3\,!}\,[s^3{}_p+2s_{3p}-3s_p.s_{2p}].$$

Putting $p=3$, $q=2$, $r=1$, we get

$$\Sigma\alpha^3\beta^2\gamma=s_1s_2s_3+2s_6-s_1s_5-s_2s_4-s_3^2.$$ **(Delhi Hon's 52)**

Note. In case m exponents become equal, then we should divide the corresponding value by m !.

Ex. 1. *If $\alpha_1, \alpha_2, \ldots \alpha_n$ be the roots of the equation*

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ \ldots+p_{n-1}x+p_n=0,$$

find the values of the following symmetric functions :—

(i) $\Sigma\alpha_1^2.\alpha_2^2$, (ii) $\Sigma\alpha_1^3\alpha_2$,

(iii) $\Sigma\alpha_1^2\alpha_2.\alpha_3$, (iv) $\Sigma\alpha_1^2.\alpha_2^2\alpha_3$.

Find the values of the above symmetric functions in terms of s_r whose value can be found in terms of the coefficients.

(i) $p_2^2-2p_1p_3+2p_4$, (ii) $p_1^2p_2-2p_2^2-p_1p_3+4p_4$,

(iii) $p_1.p_3-4p_4$, (iv) $-p_2p_3+3p_1p_4-5p_5$. **Ans.**

Above procedure will involve very lengthy simplifications and calculation. The students may however try to calculate the symmetric functions by directly multiplying two symmetric functions whose value we know directly in terms of coefficients *i.e.* for (iii), we see that

$$\Sigma\alpha_1 . \Sigma\alpha_1\alpha_2\alpha_3 = \Sigma\alpha_1{}^2\alpha_2\alpha_3 + 4\alpha_1\alpha_2\alpha_3\alpha_4 \text{ etc.}$$

Ex. 2. *If* α, β, γ *are the roots of* $x^3+px^2+qx+r=0$, *find the value of* $\Sigma\alpha^3\beta^2$. **(Pb. 58)**

Putting $p=3$ and $q=2$ in 1, we get

$$\Sigma\alpha^3\beta^2 = s_3 . s_2 - s_5 \text{ etc.}$$

CHAPTER V

SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS

§ 1. Explain Cardon's method of solving the cubic equation. **(Pb. 32, 34, 36, 38, 40, 44, 49, 53, 56 ; Delhi Hon's 52 ; Agra 53)**

Solve the equation $x^3-15x^2-33x+847=0.$

(Pb. 52 ; Agra 53)

Let the cubic equation be

$$a_0x^3+3a_1x^2+3a_2x+a_3=0 \quad \ldots \quad \ldots (1)$$

The above equation can be reduced to the form

$$z^3+3Hz+G=0 \quad \ldots \quad \ldots \quad \ldots (2)$$

where $z=a_0x+a_1$. [§ 9, P. 52]

Now let us assume that $z=u+v$; cubing both sides,

$$z^3=u^3+v^3+3uv(u+v)$$

or

$$z^3-3uvz-(u^3+v^3)=0 \quad \ldots \quad \ldots \quad \ldots (3)$$

Comparing (2) and (3), we get

$$uv=-H \ldots \quad \ldots \quad \ldots \quad \ldots (4)$$

and

$$u^3+v^3=-G.$$

$\therefore$ $u^3v^3=-H^3$ and hence u^3 and v^3 are the root of the quadratic

$$t^2+Gt-H^3=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots (5)$$

$$\therefore \quad u^3=\frac{-G+\sqrt{(G^2+4H^3)}}{2} \text{ and } v^3=\frac{-G-\sqrt{(G^2+4H^3)}}{2}.$$

Now from above we shall obtain three values of u and three of v ; our root being $u+v$ will have therefore nine possible values because each of the three values of u is to be added to any of the three values of v. But there is a limitation imposed by the relation (4), *i.e.* $uv=-H=$constant

or $$v=\frac{-H}{u}.$$

Hence the value of u and v to be combined should be such as to satisfy $uv=-H$. Now if we extract cube root of u^3, we get u, $u\omega$, $u\omega^2$ and that of v^3 gives v, $v\omega$, $v\omega^2$.

Hence the three roots of the equation are

$$u+v,\ u\omega+v\omega^2,\ u\omega^2+v\omega$$

where $\omega=\frac{-1+\sqrt{(-3)}}{2}$ and $\omega^2=\frac{-1-\sqrt{(-3)}}{2}$.

Having found z we can find the value of x by the relation $z=a_0x+a_1$,

Note— The above method will not be able to give the solution of all cubics with numerical coefficients. Only when $G^2+4H^3>0$ or $=0$ *i.e.* when the cubic has two imaginary or two equal roots [P. 77] then the values of u^3 and v^3 from 't' quadratic will be real and hence u and v can be determined but in case $G^2+4H^3<0$, *i.e.* all the roots of the cubic be real [P. 77] then the roots of 't' quadratic *i.e.* u^3 and v^3 are both imaginary and hence u and v will be cube roots of imaginary quantities which has no arithmetical meaning. Hence Cardon's method will fail to give the solution of a cubic all of whose roots are real. This is called **irreducible case of Cardon's solution.** We shall discuss it by taking numerical examples in which we shall use either trial method or trigonometrical method without the help of which Cardon's method will not give the solution. Hence we can solve the given cubic by Cardon's method when two of its roots are either equal or two imaginary.

Ex. Solve $x^3-15x^2-33x+847=0$.

Diminish the roots by 5 in order to reduce the equation to standard form and the transformed equation is

$$z^3-108z+432=0.$$

Let $z=u+v$; $\therefore$ $z^3-3uvz-(u^3+v^3)=0$

or $uv=36$ and $u^3+v^3=-432$.

$\therefore$ u^3 and v^3 are the roots of $t^2+432t+36^3=0$

or $t^2+2\cdot216.t+(6^3)^2=0$ or $(t^2+6^3)^2=0$.

$\therefore$ $t=-6^3, -6^3$ *i.e.* $u=-6$, $v=-6$.

$\therefore$ $u+v=-12$ is a root of the z-cubic which when divided by $z+12$ gives the quadratic

$$z^2-12z+36=0; \quad \therefore \quad (z-6)^2=0.$$

$\therefore$ roots of z cubic are 6, 6, -12 and hence of x cubic are 11, 11, -7 **(two roots equal).**

§ 2. Discuss the nature of the roots of the cubic.

Nature of the roots of the cubic.

Discriminant. As already defined at P. 80 § 13, the expression G^2+4H^3 is called the discriminant of the cubic and its vanishing is the necessary and sufficient condition for the cubic to have two roots equal as will be clear from what follows.

The nature of the roots of the given cubic is not changed by the linear transformation. $z=a_0x+a_1$ and hence we shall discuss the nature of the roots of the equation $z^3+3Hz+G=0$ whose roots we have found to be $u+v$, $u\omega+v\omega^2$, $u\omega^2+v\omega$ say z_1, z_2, z_3,

or $$u+v,\ u.\frac{-1+\sqrt{(-3)}}{2}+v.\frac{-1-\sqrt{(-3)}}{2}$$

and $$u.\frac{-1-\sqrt{(-3)}}{2}+v.\frac{-1+\sqrt{(-3)}}{2}.$$

or $$u+v,\ \frac{-1}{2}(u+v)\pm\frac{1}{2}(u-v)\,i\sqrt{3}\ldots\ \ldots\ \ldots(1)$$

where u^3 and v^3 are the roots of $t^2+Gt-H^3=0$.

Case 1. $\mathbf{G^2+4H^3=0}$**.** In this case the roots of t quadratic are equal and hence u and v are equal.

$\therefore$ $u-v=0$ and from (1) we obtain that two of the roots of z cubic are equal, each being $\frac{-1}{2}(u+v)$ and consequently **two roots of x cubic are equal.**

Converse. Two roots of x cubic and hence of z cubic are equal; then $G^2+4H^3=0$.

Suppose $z_1=z_2$; $\therefore$ $u+v=u\omega+v\omega^2$ or $u(1-\omega)=-v(1-\omega^2)$ or $u=-v(1+\omega)$ or $u=v\omega^2$ $\because$ $1+w=-\omega^2$.

$\therefore$ $u^3=v^3\omega^6$ or $u^3=v^3$ *i.e.* the roots of t quadratic are equal and hence $G^2+4H^3=0$.

Case 2. $\mathbf{G=0, H=0}$**.** In this case the z cubic and hence the x cubic has evidently **three equal roots.**

Converse. Three roots of x cubic and hence of z cubic are equal; then $G=0$, $H=0$.

If $z_1=z_2$, then we have proved $u=v\omega^2$.

If $z_2=z_3$, then similarly $u=v$.

$\therefore$ $u=u\omega^2$ or $u(1-\omega^2)=0$ *i.e.* $u=0$ and hence $v=0$.

$\therefore$ u^3 and v^3 are also zero *i.e.* the roots of t quadratic are both zero and therefore $G=0$, $H=0$.

Case 3. $\mathbf{G^2+4H^3>0}$**.** In this case the roots of the quadratic are real *i.e.* u^3 and v^3 are real and hence one of the values of u and v is real *i.e.* $u+v$ is real or z_1 **is real and the other two are imaginary.**

Converse—The given cubic and hence z cubic has one root real and two imaginary; then $G^2+4H^3 > 0$.

In case two roots are to be imaginary, then from (1), we conclude that $u+v$=real and $u-v$ is real.

$\therefore$ both u and v and hence u^3, v^3 the roots of t quadratic are real.

$$\therefore \quad G^2+4H^3 > 0.$$

Case 4. $G^2+4H^3 < 0$. In this case the roots of t quadratic are imaginary conjugate, *i.e.* u^3 and v^3 and hence both u and v are conjugate complex *i.e.* if $u=a+ib$, then $v=a-ib$.

$$\therefore \quad u-v=2ib \text{ and } u+v=2a.$$

Substituting these values in (1), we find that the roots of z cubic and hence of **x cubic are all real.**

Converse—The given cubic has all roots real : then $G^2+4H^3 < 0$.

In case all the roots of x cubic and hence of z cubic are to be real then from (1), $u+v$=real and $u-v$=pure imaginary *i.e.* u and v are conjugate complex.

$\therefore$ u^3, v^3 the roots of t quadratic are also conjugate complex.

$$\therefore \quad G^2+4H^3 < 0.$$

Ex. 1. (a) *Solve the following cubics* :—

$$x^3-12x^2-6x-10=0, \qquad \textbf{(Pb. 32)}$$

$$4+(u+v),\ 4+(u\omega+v\omega^2),\ 4+(u\omega^2+v\omega),$$

where $u=-3\sqrt[3]{4},\ v=-3\sqrt[3]{2}.$ **Ans.**

(b) $x^3+6x^2+9x+4=0.$ **(Pb. 34)**

$-4,\ -1,\ -1.$ **Ans.**

(c) $x^3-15x^2-357x+5491=0.$ **(Pb. 38)**

z cubic is $z^3-432z+3456=0$ and t quadratic is

$$t^2+3456t+144^3=0 \quad \text{or} \quad (t+12^3)^2=0.$$

$-19, 17, 17.$ **Ans.**

(d) $28x^3-9x^2+1=0.$ **(Pb. 40, 46)**

Put $x=\frac{1}{z}$ etc. $-\frac{1}{4}, \frac{2\pm i\sqrt{3}}{7}.$

After finding one root of z cubic and hence of x cubic divide it by the corresponding factor and find the remaining two roots from the quadratic.

(e) $x^3+x^2-16x+20=0.$ **(Pb. 49)**

$2, 2, -5$ (2 is easily seen to be a root, etc.) **Ans.**

(f) $x^3+72x-1720=0.$

$10, -5\pm i7\sqrt{3}.$ **Ans.**

(g) $2x^3+3x^2+3x+1=0.$ **(Pb. 41 Supp.)**

Multiply the roots by 2 and the transformed equation is $y^3+3y^2+6y+4=0$. Now diminish the roots by -1 etc.

$-\frac{1}{2}, \frac{1}{2}(-1\pm i\sqrt{3}).$ **Ans.**

(h) $27x^3+54x^2+198x-73=0.$

$na_0h+a_1=0$ gives $h=-\frac{2}{3}$. Diminish the root by $-\frac{2}{3}$ and the transformed equation is $z^3+54z-189=0$.

$\frac{1}{3}, \frac{-7\pm i9\sqrt{3}}{6}.$ **Ans.**

(i) $x^3-21x-344=0.$ **(Pb. 56)**

t quadratic is $t^2-344t+343=0$.

$8, (-4\pm i3\sqrt{3}).$ **Ans.**

Cubic having all roots real.

Ex. 2. *Prove that the roots of the equation* $x^3-3x+1=0$ *are* $2\cos\frac{2\pi}{9}, 2\cos\frac{8\pi}{9}, 2\cos\frac{14\pi}{9}.$ **(Pb. 53)**

Let $x=u+v$. $\therefore$ $x^3-3uvx-(u^3+v^3)=0$.

$\therefore$ $uv=1$ and $u^3+v^3=-1$ and hence u^3 and v^3 are the roots of

$$t^2+t+1=0 \quad \text{or} \quad t=\frac{-1\pm i\sqrt{3}}{2}.$$

Let $u^3=-\frac{1}{2}+i\frac{\sqrt{3}}{2}=r(\cos\theta+i\sin\theta)$.

$$\therefore \quad r=1 \text{ and } \tan\theta=-\sqrt{3}, \ \therefore\ \theta=120=\frac{2\pi}{3}$$

or $$u^3=\cos(2n\pi+\theta)+i\sin(2n\pi+\theta).$$

$$\therefore \quad u=\left[\cos\left(2n\pi+\frac{2\pi}{3}\right)+i\sin\left(2n\pi+\frac{2\pi}{3}\right)\right]^{1/3}$$

$$=\cos\frac{2\pi(3n+1)}{9}+i\sin\frac{2\pi(3n+1)}{9}$$

(De Moivre's Theorem).

The three values of u are obtained by putting $n=0, 1, 2$.

$$\therefore \quad u=\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9},$$

$$\cos\frac{8\pi}{9}+i\sin\frac{8\pi}{9} \text{ and } \cos\frac{14\pi}{9}+i\sin\frac{14\pi}{9}.$$

$$\therefore \quad v=\frac{1}{u} \quad \because\ uv=1$$

and $$\frac{1}{\cos\theta+i\sin\theta}=(\cos\theta+i\sin\theta)^{-1}=\cos\theta-i\sin\theta.$$

$$\therefore \quad v=\cos\frac{2\pi}{9}-i\sin\frac{2\pi}{9},$$

$$\cos\frac{8\pi}{9}-i\sin\frac{8\pi}{9}, \cos\frac{14\pi}{9}-i\sin\frac{14\pi}{9}.$$

Hence the roots are $u+v$, where the values of u and v are to be chosen so as to satisfy $uv=1$. Hence the roots are

$$2\cos\frac{2\pi}{9}, 2\cos\frac{8\pi}{9}, 2\cos\frac{14\pi}{9}.$$

(b) *Solve the cubic $x^3-6x+4=0$.*

$$2\sqrt{2}\cos\frac{\pi}{12},\ 2\sqrt{2}\cos\frac{3\pi}{4},\ 2\sqrt{2}\cos\frac{17\pi}{12},$$

i.e. $$\sqrt{3}-1,\ -2,\ -\sqrt{3}+1.$$

(c) *Show that the roots of the cubic*

$$x^3-3a^2x-2a^3\cos 3A=0$$

are $$2a\cos A,\ 2a\cos(120^\circ\pm A).$$

Let $x=u+v$. $\therefore$ $x^3-3uvx-(u^3+v^3)=0$.

$$\therefore\quad uv=a^2,\ \text{and}\ u^3+v^3=2a^3\cos 3A.$$

Hence u^3, v^3 are the roots of $t^2-2a^3\cos 3A+a^6=0$.

$$\therefore\quad t=\frac{2a^3\cos 3A\pm\sqrt{(4a^6\cos^2 3A-4a^6)}}{2}$$

$$=a^3[\cos 3A\pm i\sin 3A]$$
$$=a^3[\cos(2n\pi+3A)\pm i\sin(2n\pi+3A)].$$

Taking cube root the values of u and v are found out to be

$$a\left[\cos\frac{(2n\pi+3A)}{3}\pm i\sin\frac{2n\pi+3A}{3}\right],$$

where $n=0, 1, 2$ (by De Moivre's Theorem).

$$\therefore\quad u=a(\cos A+i\sin A)$$
$$=a\left[\cos\left(\frac{2\pi}{3}+A\right)+i\sin\left(\frac{2\pi}{3}+A\right)\right],$$
$$=a\left[\cos\left(\frac{4\pi}{3}+A\right)+i\sin\left(\frac{4\pi}{3}+A\right)\right],$$

and $$v=a(\cos A-i\sin A)$$
$$=a\left[\cos\left(\frac{2\pi}{3}+A\right)-i\sin\left(\frac{2\pi}{3}+A\right)\right]$$
$$=a\left[\cos\left(\frac{4\pi}{3}+A\right)-i\sin\left(\frac{4\pi}{3}+A\right)\right].$$

Hence the roots are the values of $u+v$, *i.e.*,

$$2a\cos A,\ 2a\cos\left(\frac{2\pi}{3}+A\right),\ 2a\cos\left(\frac{4\pi}{3}+A\right)$$

or $$2a\cos A,\ 2a\cos(120+A),\ 2a\cos\left\{2\pi-\left(\frac{4\pi}{3}+A\right)\right\}$$

or $$2a\cos A,\ 2a\cos(120+A),\ 2a\cos(120-A)$$

or $$2a\cos A,\ 2a\cos(120\pm A).$$

(d) *Find the roots of the cubic*

$$8a^3x^3-6ax+2\sin 3A=0.$$

Proceeding as in part (c), u^3 and v^3 are the roots of

$$64a^6t^2+16a^3t\sin 3A+1=0.$$

$$\therefore\quad t=\frac{1}{8a^3}[-\sin 3A\pm i\cos 3A].$$

$$u^3,v^3=\frac{1}{8a^3}\left[\cos\left(\frac{\pi}{2}+3A\right)+i\sin\left(\frac{\pi}{2}+3A\right)\right].$$

$$x=u+v=\frac{1}{2a}\left[\cos\left(2n\pi+\frac{\pi}{2}+3A\right)+i\sin\left(2n\pi+\frac{\pi}{2}+3A\right)\right]^{1/3}$$

$$+\frac{1}{2a}\left[\cos\left(2n\pi+\frac{\pi}{2}+3A\right)-i\sin\left(2n\pi+\frac{\pi}{2}+3A\right)\right]^{1/3},$$

where $n=0,1,2$

$$=\frac{1}{a}\cos\left(\frac{\pi}{6}+A\right),\ \frac{1}{a}\cos\left(\frac{5\pi}{6}+A\right),\ \frac{1}{a}\cos\left(\frac{3\pi}{2}+A\right)$$

$$=\frac{1}{a}\cos(30+A),\ \frac{1}{a}\cos(150+A),\ -\frac{1}{a}\sin A$$

or $$\frac{1}{a}\cos\{90-(30+A)\},\ \frac{1}{a}\cos\{90+60+A\},\ -\frac{1}{a}\sin A$$

or $$\frac{1}{a}\sin(60-A),\ -\frac{1}{a}\sin(60+A),\ -\frac{1}{a}\sin A.$$

Ex. 3. (a) $x^3-7x+6=0.$

Proceeding as usual, t quadratic is

$$27t^2+162t+343=0.$$

$$\therefore\quad u^3,v^3=\frac{81\pm30\sqrt{(-3)}}{-27}.$$

Always make the denominator of the values of u^3, v^3 a perfect cube. If it is not so, then multiply above and below. Here it is a perfect cube.

Now in order to find u and v, we must know the cube root of $81\pm30\sqrt{(-3)}$ which suppose is $a+b\sqrt{(-3)}$.

$$\therefore\quad 81+30\sqrt{(-3)}=\{a+b\sqrt{(-3)}\}^3$$
$$=a^3-9ab^2+3\sqrt{(-3)}\,(a^2b-b^3).$$
$$\therefore\quad a\,(a^2-9b^2)=81 \text{ and } b\,(a^2-b^2)=10.$$

∴ a should be a factor of 81 and b that of 10. By trial we find that $a=-3$ and $b=2$.

$$\therefore\quad u^3=\left\{\frac{-3+2\sqrt{(-3)}}{-3}\right\}^2 \quad\text{or}\quad u=\frac{-3+2\sqrt{(-3)}}{-3} \quad\text{and}$$

hence $v=\dfrac{-3-2\sqrt{(-3)}}{-3}$. Adding, we get $u+v=2$.

∴ $x=2$ is a root and dividing the cubic by $x-2$, we get the quadratic $(x^2+2x-3)=0$ and other roots are -3, 1.

(b) *Solve part (a) and (b) by the trial method.*

(c) $x^3+3x-14=0$.

Proceeding as above,

$$u^3,\ v^3=7\pm5\sqrt{2}.$$

Let $7+5\sqrt{2}=(a+b\sqrt{2})^3$

$$=(a^3+6b^2a)+\sqrt{2}\,(2b^3+3a^2b);$$
$$\therefore\quad a\,(a^2+6b^2)=7 \text{ and } b\,(2b^2+3a^2)=5.$$

From above we find that a should be a factor of 7 and b that of 5. By trial we observe that $a=1$, $b=1$.

∴ $u^3=(1+\sqrt{2})^3$ or $u=1+\sqrt{2}$ and hence $v=1-\sqrt{2}$.

∴ $x=u+v=2$ is a root, and dividing the given cubic by $(x-2)$, we get the quadratic $x^2+2x+7=0$.

$$\therefore\quad x=-1\pm i\sqrt{6}.$$

Note—Even though the equation is not having all the roots real but still we had to apply trial method.

(*d*) $x^3-3x^2-10x+24=0.$

2, -3, 4. **Ans.**

(*e*) $x^3-26x+60=0.$

$-6, 3\pm i.$ **Ans.**

Use trial as in part (*c*), or otherwise -6 seems to be a root.

§ 3. Expressing the cubic as sum or difference of two cubes.

Show that if p *and* q $(p\neq q)$ *be the roots of* $(a_0x+a_1)\times(a_2x+a_3)=(a_1x+a_2)^2$, *then the cubic equation* $a_0x^3+3a_1x^2+3a_2x+a_3=0$ *can be reduced to the form* $A\,(x-p)^3+B\,(x-q)^3$. **(Agra 39, 50, 60; Delhi Hon's 48)**

$$a_0x^3+3a_1x^2+3a_2x+a_3\equiv A\,(x-p)^3+B\,(x-q)^3 \quad \ldots(1)$$

Comparing the coefficients of like powers of x, we get

$$A+B=a_0 \quad \ldots \ldots \ldots \ldots(2)$$

$$-(Ap+Bq)=a_1 \quad \ldots \ldots \ldots \ldots(3)$$

$$(Ap^2+Bq^2)=a_2 \quad \ldots \ldots \ldots \ldots(4)$$

$$-(Ap^3+Bq^3)=a_3 \quad \ldots \ldots \ldots \ldots(5)$$

Now multiply (2), (3) and (4) by p and adding respectively to (3), (4) and (5), we get

$$B.(p-q)=a_0p+a_1 \ldots \ldots \ldots \ldots(6)$$

$$-Bq\,(p-q)=a_1p+a_2 \quad \ldots \ldots \ldots(7)$$

$$Bq^2\,(p-q)=a_2p+a_3 \quad \ldots \ldots \ldots(8)$$

From (6), (7) and (8), we observe that

$$(a_0p+a_1)\,(a_2p+a_3)=(a_1p+a_2)^2 \ldots \ldots \ldots(9)$$

Similarly multiply (2), (3), and (4) by q and add respectively to (3), (4) and (5), and proceeding as above, we get

$$(a_0q+a_1)\,(a_2q+a_3)=(a_1q+a_2)^2 \ldots \ldots \ldots(10)$$

and (9) and (10) show that p and q are the roots of

$$(a_0x+a_1)\,(a_2x+a_3)=(a_1x+a_2)^2 \ldots \ldots \ldots(11)$$

or $$(a_0a_2-a_1^2)\,x^2+(a_0a_3-a_1a_2)\,x+(a_1a_3-a_2^2)=0 \quad \ldots(12)$$

Having found p and q from above we can find A and B from (2) and (3) and the given cubic can be put as

$$A\,(x-p)^3=-B\,(x-q)^3$$

or $$\left(\frac{x-p}{x-q}\right)^3=\left(\frac{-B}{A}\right).$$

Taking cube-root of both sides, we can find one of the values of x and then depress the equation by one dimension and the remaining two roots can be found from the **quadratic.**

One condition has however to be seen and is that if roots of (12) *i. e.* the values of p and q are imaginary and hence of A, B as such, then we shall not be able to extract the cube-roots of complex quantities. Hence in order that the cubics may be solved by the above method, the roots of quadratic (12) should be real, the condition for which is

$$B^2-4AC>0,$$

i.e., $(a_0a_3-a_1a_2)^2-4\,(a_1a_3-a_2^2)\,(a_0a_2-a_1^2)>0.$

Hessian :—

Note— If we make the given cubic homogeneous by introducing y where y is to be put equal to (1) afterwards.

i.e., $$f(x,y)=a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3=0,$$

then $$\frac{\partial f}{\partial x}=3(a_0x^2+2a_1xy+a_2y^2),$$

$$\frac{\partial^2 f}{\partial x^2}=6(a_0x+a_1y), \qquad \frac{\partial^2 f}{\partial x\,\partial y}=6(a_1x+a_2y)$$

and $$\frac{\partial^2 f}{\partial x^2}=6(a_2x+a_3y).$$

Putting $y=1$, we find that the quadratic equation whose roots are p and q is nothing but

$$\frac{\partial^2 f}{\partial x^2}\cdot\frac{a^2 f}{dy^2}=\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2.$$

The above equation is called the **Hessian of the cubic** and its roots should be real if the cubic is to be solved by expressing it as the sum or difference of two cubes.

Alternative method of finding the quadratic whose roots are p and q.

Let p and q be the roots of the equation

$$x^2+\lambda x+\mu=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots(13)$$

$$\therefore \quad p^2+\lambda p+\mu=0 \quad \text{and} \quad q^2+\lambda q+\mu=0.$$

Now multiply (4) by (1), subtract (3) multiplied by λ and add (2) multiplied by μ ; we get

$$A(p^2+\lambda p+\mu)+B(q^2+\lambda q+\mu)=a_2-a_1\lambda+a_0\mu.$$

$$\therefore \quad a_2-a_1\lambda+a_0\mu=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots(14)$$

Similarly, multiply (5) by (1), subtract (4) multiplied by λ and add (2) multiplied by μ ; we get

$$a_3-a_2\lambda+a_1\mu=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots(15)$$

Eliminating 1, λ, μ from (13), (14), (15), we get the quadratic whose roots are p and q in determinant form as

$$\begin{vmatrix} x^2 & x & 1 \\ a_2 & -a_1 & a_0 \\ a_3 & -a_2 & a_1 \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} x^2 & -x & 1 \\ a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 \end{vmatrix}=0$$

or

$$\begin{vmatrix} 1 & -x & x^2 \\ a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{vmatrix}=0$$

or

$$x^2(a_0a_2-a_1^2)+x(a_0a_3-a_1a_2)+(a_1a_3-a_2^2)=0$$

which is same as (12).

Note :—In practice try to form the Hessian of the given cubic and having found p and q from the Hessian find the values of A and B from (1) and (2) or find the values of p and q from the determinant

$$\begin{vmatrix} 1 & -x & x^2 \\ a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0.$$

Ex. 1. *Solve the equation* $2x^3+3x^2-21x+19=0$ *by expressing it as the sum or difference of two cubes.* **(Agra 39)**

Let $2x^3+3x^2-21x+19 \equiv A(x-p)^3-B(x-q)^3$ ... ... (1)

p and q will be the roots of $\begin{vmatrix} 1 & -x & x^2 \\ a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{vmatrix}$

where $a_0=2$, $a_1=1$, $a_2=-7$, and $a_3=19$.

Putting the values of a_0, a_1, a_2, a_3 and expanding, we get the quadratic

$$x^2-3x+2=0 \text{ or } (x-1)(x-2)=0. \therefore p=1, q=2.$$

Again comparing the coefficients of x^3 and x^2 in (1), we get $A-B=2$ and $Ap-Bq=-1$.

Putting the values of p and q and solving, we get $A=5$ and $B=3$.

$$\therefore \quad 5(x-1)^3=3(x-2)^3$$

or
$$\frac{x-1}{x-2}=\frac{\sqrt[3]{3}}{\sqrt[3]{5}}k,$$

where $k=1$, ω, ω^2 and $k^3=1$.

$$x(\sqrt[3]{5}-k\sqrt[3]{3})=\sqrt[3]{5}-2k\sqrt[3]{3}.$$

$$\therefore \quad x=\frac{(5)^{1/3}-2k(3)^{1/3}}{(5)^{1/3}-k(3)^{1/3}}.$$

Multiply above and below by a^2+ab+b^2 so that D^r becomes a^3-b^3.

$$\therefore \quad x=\frac{[(5)^{1/3}-2k(3)^{1/3}][(25)^{1/3}+k(5)^{1/3}(3)^{1/3}+k^2(9)^{1/3}]}{5-3}$$

Multiplying and putting $k^3=1$,

$$x=\frac{5-k\sqrt[3]{75}-k^2\sqrt[3]{45}-6}{2}.$$

Now put $k=1,\ \omega,\ \omega^2$ for the three roots which are

$$\tfrac{1}{2}\left(-1-\sqrt[3]{75}-\sqrt[3]{45}\right),\quad \tfrac{1}{2}\left(-1-\omega\sqrt[3]{75}-\omega^2\sqrt[3]{45}\right)$$

and $\tfrac{1}{2}\left(-1-\omega^2\sqrt[3]{75}-\omega\sqrt[3]{45}\right)$, $\because\ \omega^3=1$.

Ex. 2. *Solve the equation* $x^3+3x^2-27x+104=0$ *by expressing it as the sum or difference of two cubes.* **(Agra 60)**

Let $x^3+3x^2-27x+104=A(x-p)^3-B(x-q)^3$ ... ...(1)

p and q will be the roots of

$$\begin{vmatrix} 1 & -x & x^2 \\ 1 & 1 & -9 \\ 1 & -9 & 104 \end{vmatrix}=0$$

or $10x^2-113x-23=0$ or $(5x+1)(2x-23)=0$.

$$\therefore\quad p=-\tfrac{1}{5} \text{ and } q=\tfrac{23}{2}.$$

Comparing the coefficients of x^3 and x^2 in (1), we get

$$A-B=1,$$

$$Ap-Bq=-1.$$

or $-\frac{1}{5}A-\frac{23}{2}B=-1$.

Solving these two equations, we get

$$A=\tfrac{125}{117} \text{ and } B=\tfrac{8}{117}.$$

Hence $\dfrac{x-p}{x-q}=\left(\dfrac{B}{A}\right)^{1/3}=\left(\dfrac{8}{125}\right)^{1/3}=\dfrac{2}{5}$

or $3x=5p-2q=-24$. $\therefore\ x=-8$.

Dividing the given equation by $x+8$, we get the quadratic $x^2-5x+13=0$ giving the other roots as $\dfrac{5\pm i3\sqrt{3}}{2}$.

(b) *Solve the cubic* $152x^3-60x^2-606x-485=0$ *by expressing it as the sum or difference of two cubes.*

$$\frac{5}{2},\ \frac{-40\pm i7\sqrt{7}}{38}.$$ **Ans.**

(c) *By expressing the equation*

$$35x^3+339x^2+1113x+1241=0$$

as the sum of two cubes, find its roots.

$$-\frac{17}{5},\ \frac{-22\pm3\sqrt{3}i}{7}.$$ **Ans.**

Ex. 3. *Prove that if a cubic has two roots equal, then its Hessian has also two equal roots in common with the cubic and conversely if the Hessian be a perfect square i.e. having two equal roots then it is a factor of the cubic which has two equal roots.*

Let the cubic be $x^3+3Hx+G=0$. If it has two equal roots then we know that $G^2+4H^3=0$... ... ...(1) (P. 81)

The Hessian of the given cubic is

$$\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2=0 \text{ where } y=1,$$

or in determinant form

$$\begin{vmatrix} 1 & -x & x^2 \\ a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{vmatrix}=0,\quad \begin{matrix} a_0=1,\ a_1=0 \\ a_2=H,\ a_3=G \end{matrix}$$

or $$\begin{vmatrix} 1 & -x & x^2 \\ 1 & 0 & H \\ 0 & H & G \end{vmatrix}=0 \text{ i.e. } Hx^2+Gx-H^2=0\ldots\ldots(2)$$

Now if $G^2+4H^3=0$; then the discriminant of the quadratic Hessian (2) vanishes and it has two equal roots.

Now we have to prove that the equal roots of (1) and (2) are common. Divide the cubic by Hessian and we get the quotient as $x-\frac{G}{H}$ and remainder $\frac{G^2+4H^3}{H}x$. The remainder will be zero as $G^2+4H^3=0$, showing that the Hessian which has equal roots divides the cubic without remainder and as such it is its factor.

Conversely :—If the Hessian be a perfect square, we have equal roots; then $G^2+4H^3=0$ and with the given condition Hessian becomes a factor of the cubic and as such it will also have two equal roots in common with that of the Hessian.

Ex. 4. (a) *If the cubic equation* $a_0x^3+3a_1x^2+3a_2x+a_3=0$ *has two roots equal to* α, *prove that*

$$-\alpha=\frac{H_2}{H_1}=\frac{H_1}{H}$$

where $H=a_0a_2-a_1^2$, $2H_1=a_0a_3-a_1a_2$, $H_2=a_1a_3-a_2^2$.

(Agra 38, 46; Pb. 43)

We know that if a cubic has two roots equal, then its Hessian has also two equal roots in common with the cubic. The Hessian of the given cubic is

$$(a_0a_2-a_1^2)\, x^2+(a_0a_3-a_1a_2)\, x+(a_1a_3-a_2^2)=0$$

or $$Hx^2+2H_1x+H_2=0 \quad \ldots \quad \ldots \quad \ldots(1)$$

Since it has equal roots, $\therefore\ 4H_1^2-4HH_2=0$

or $$H_1^2=HH_2 \ldots \quad \ldots \quad \ldots \quad \ldots(2)$$

Now each of the equal roots is common with that of the cubic and hence it is also α.

$$\therefore\ 2\alpha=\text{sum of the roots of (1)}=-\frac{2H_1}{H}$$

or $$-\alpha=\frac{H_1}{H} \text{ and also }=\frac{H_2}{H_1} \text{ from (2).} \qquad \text{Hence proved.}$$

Alternative method. (Pb. 56)

We know that if $f(x)=0$ has two roots each equal to α, then α must also be a root of $f'(x)=0$.

$$\therefore\ a_0\alpha^3+3a_1\alpha^2+3a_2\alpha+a_3=0 \ldots \quad \ldots \quad \ldots(1)$$

$$a_0\alpha^2+2a_1\alpha+a_2=0 \ldots \quad \ldots \quad \ldots \quad \ldots(2)$$

Multiplying (2) by α and subtracting from (1), we get

$$a_1\alpha^2+2a_2\alpha+a_3=0 \ldots \quad \ldots \quad \ldots \quad \ldots(3)$$

From (2) and (3), we have

$$\frac{\alpha^2}{a_1a_3-a_2^2}=\frac{2\alpha}{a_1a_2-a_0a_3}=\frac{1}{a_0a_2-a_1^2}$$

or

$$\frac{\alpha^2}{H_2}=\frac{2\alpha}{-2H_1}=\frac{1}{H};$$

$$\therefore\quad -\alpha=\frac{H_2}{H_1}=\frac{H_1}{H}.$$

Proved.

(b) *If* $x^3+3px^2+3qx+r$ *and* $x^2+2px+q$ *have a common factor show that*

$$4\,(p^2-q)\,(q^2-pr)=(pq-r)^2. \qquad \textbf{(Agra '42)}$$

If $x-\alpha$ be the common factor then α is a root of

$$f\,(x)=x^3+3px^2+3qx+r=0$$

and

$$x^2+2px+q=0. \qquad \text{which is } f'\,(x).$$

$$\therefore\quad \alpha^3+3p\alpha^2+3q\alpha+r=0 \;\ldots\quad \ldots\quad \ldots(1)$$

$$\alpha^2+2p\alpha+q=0 \;\ldots\quad \ldots\quad \ldots\quad \ldots(2)$$

Multiplying (2) by α and subtracting from (1), we get

$$p\alpha^2+2q\alpha+r=0\ldots\quad \ldots\quad \ldots\quad \ldots(3)$$

Solving (2) and (3) by the method of cross-multiplication,

$$\frac{\alpha^2}{2pr-2q^2}=\frac{\alpha}{pq-r}=\frac{1}{(2q-2p^2)};$$

$$\therefore\quad 2\,(pr-q^2)\,2\,(q-p^2)=(pq-r)^2$$

or

$$4\,(q^2-pr)\,(p^2-q)=(pq-r)^2.$$

Ex. 5. *If the cubic*

$$ax^3+3bx^2+3cx+d\equiv l\,(x+\theta)^3+m\,(x+\phi)^3,$$

show that θ *and* ϕ *are the roots of the quadratic equation*

$$\begin{vmatrix} 1 & x & x^2 \\ a & b & c \\ b & c & d \end{vmatrix}=0. \qquad \textbf{(Agra 36, 47)}$$

See § 3.

Ex. 6. *Prove that if a cubic has three roots equal, its Hessian quadratic vanishes identically and conversely.*

The cubic has three roots equal if $G=0$, $H=0$ [§ 1] and hence the Hessian quadratic $Hx^2+Gx-H^2=0$ of the cubic $x^3+3Hx+G=0$ vanishes identically

Ex. 7. *Find the relation between q and r in order that the equation $x^3+qx+r=0$ may be put in the form*

$$x^4=(x^2+ax+b)^2.$$

Hence solve the equation $8x^3-36x+27=0$. **(Agra 42)**

$$x^4=x^4+2ax^3+(a^2+2b)x^2+2abx+b^2$$

or $$x^3+\frac{a^2+2b}{2a}x^2+bx+\frac{b^2}{2a}=0,$$

Comparing with the given cubic, we get

$$a^2+2b=0,\quad b=q,\text{ and }\frac{b^2}{2a}=r;\quad \therefore\quad a=\frac{b^2}{2r}=\frac{q^2}{2r}\,.$$

Putting the values of a and b in $a^2+2b=0$, we get the required relation as $8r^2+q^3=0$.

$$8x^3-36x+27=0$$

or $$x^3-\tfrac{9}{2}x+\tfrac{27}{8}=0.$$

Its coefficients satisfy the relation $8r^2+q^3=0$.

$$\therefore\quad b=q=-\tfrac{9}{2}\text{ and }a=\frac{q^2}{2r}=3.$$

Hence it can be put in the form

$$x^4=\left(x^2+3x-\frac{9}{2}\right)^2$$

or $x^2+3x-\frac{9}{2}=\pm x^2$, etc. $x=\frac{3}{2},\ -\frac{3}{4}(1\pm\sqrt{3})$. **Ans.**

Ex. 8. *Find the condition that the cubic $ax^3+3bx^2+3cx+d$ may be capable of being written under the form*

$$l(x-\alpha_1)^3+m(x-\beta_1)^3+n(x-\gamma_1)^3$$

where α_1, β_1, γ_1 are the roots of the cubic $a_1x^3+3b_1x^2+3c_1x+d_1=0$.

Comparing the coefficients of like powers of x, we get

$$l+m+n=a \quad \ldots \quad \ldots \quad \ldots (1)$$

$$l\alpha_1+m\beta_1+n\gamma_1=-b \quad \ldots \quad \ldots \quad \ldots (2)$$

$$l\alpha_1^2+m\beta_1^2+n\gamma_1^2=c \quad \ldots \quad \ldots \quad \ldots (3)$$

$$l\alpha_1^3+m\beta_1^3+n\gamma_1^3=-d \quad \ldots \quad \ldots \quad \ldots \ (4)$$

Multiply (4), (3), (2) and (1) by a_1, $3b_1$, $3c_1$ and d_1 respectively and adding, we get

$$l\underset{0}{(a_1\alpha_1^3+3b_1\alpha_1^2+3c_1\alpha_1+d_1)}+m\underset{0}{(\ )}+n\underset{0}{(\ )}$$

$$=-da_1+3b_1c-3c_1b+ad_1$$

or $\quad (ad_1-a_1d)=3(bc_1-b_1c)$ is the required condition.

Ex. 9. *If* $ax^3+3bx^2+3cx+d+k(x-r)^3$ *be a perfect cube, prove that*

$$(ac-b^2)r^2+(ad-bc)r+(bd-c^2)=0$$

$$\phi(x)=(a+k)x^3+3x^2(b-kr)+3x(c+kr^2)+(d-kr^3).$$

If $\phi(x)$ be a perfect cube then, $\phi(x)=0$ should have three equal roots say each equal to α.

$$\Sigma\alpha=3\alpha=3\frac{kr-b}{a+k},\ \Sigma\alpha\beta=3\alpha^2=3\frac{(c+kr^2)}{a+k},\ \alpha\beta\gamma=\alpha^3=\frac{kr^3-d}{a+k}.$$

Now $(\alpha^2)^2=\alpha.\alpha^3$. Putting the values of α, α^2, α^3 and simplifying, we get

$$bd-c^2=k(2cr^2+br^3+dr) \quad \ldots \quad \ldots \quad \ldots \ (1)$$

$$\alpha^2\times\alpha=\alpha^3.$$

Proceeding as above, we get

$$ad-bc=k(ar^3+br^2-cr-d)\ldots \quad \ldots \quad \ldots \ (2)$$

and $\quad (\alpha)^2=\alpha^2$;

$$\therefore \quad ac-b^2=k(-2br-ar^2-c) \quad \ldots \quad \ldots \quad \ldots \ (3).$$

Multiplying (1), (2), (3) by 1, r and r^2 respectively and adding, we get the required condition.

Solutions of Biquadratics.

§ 4. Euler's solution of the biquadratic.

(Agra 34, 38, 41)

Let the biquadratic be $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ which can be reduced to the form

$$z^4+6Hz^2+4Gz+(a_0^2I-3H^2)=0 \quad \ldots \quad \ldots \ (1)$$

where $z=a_0x+a_1$.

Let us assume that $z=\sqrt{p}+\sqrt{q}+\sqrt{r}$ ... ...(2)

Squaring, we get

$$[z^2-(p+q+r)]=2(\sqrt{p}\sqrt{q}+\sqrt{q}\sqrt{r}+\sqrt{r}\sqrt{p}).$$

Squaring again, we get

$$z^4-2z^2(p+q+r)+(p+q+r)^2$$
$$=4[pq+qr+rp+2\sqrt{p}\sqrt{q}\sqrt{r}(\sqrt{p}+\sqrt{q}+\sqrt{r})]$$

or $$z^4-(2\Sigma p)z^2-8z\sqrt{p}\sqrt{q}\sqrt{r}+(\Sigma p)^2-4\Sigma pq=0 \ldots \quad \ldots(3)$$

Comparing the coefficients of like powers of z in (1) and (3), we get

$$\Sigma p=-3H,\ \sqrt{p}\sqrt{q}\sqrt{r}=-\frac{G}{2};\quad \therefore\quad pqr=\frac{G^2}{4}$$

and $$(\Sigma p)^2-4\Sigma pq=a_0^2I-3H^2 \text{ or } \Sigma pq=3H^2-\frac{a_0^2I}{4}.$$

Hence the equation whose roots are p, q, r is

$$t^3-(\Sigma p)t^2+(\Sigma pq)t-pqr=0$$

or $$t^3+3Ht^2+\left(3H^2-\frac{a_0^2I}{4}\right)t-\frac{G^2}{4} \ldots \quad \ldots \quad \ldots(4)$$

(Remember)

The above equation is called **Euler's cubic.**

Now we know that $G^2+4H^3=a_0^2(HI-a_0J)$;

$$\therefore\quad \frac{-G^2}{4}=H^3-\frac{a_0^2HI}{4}+\frac{a_0^3J}{4}.$$

Hence from (4), we get

$$t^3+3Ht^2+3H^2t-\frac{a_0^2I}{4}t+H^3-\frac{a_0^2HI}{4}+\frac{a_0^3J}{4}=0$$

or $$(t+H)^3-\frac{a_0^2I}{4}(t+H)+\frac{a_0^3J}{4}=0 \ldots \quad \ldots \quad \ldots(5)$$

Now putting $t+H=a_0^2\theta$, we get

$$a_0^6\theta^3-\frac{a_0^4I}{4}\theta+\frac{a_0^3J}{4}=0$$

or $$4.a_0^3\theta^3-Ia_0\theta+J=0 \quad \ldots \quad \ldots \quad \ldots(6)$$

(Remember)

Equation (6) is called **reducing cubic** of the biquadratic equation.

Note. We have seen that G occurs in even powers in Euler's cubic and hence if we had the biquadratic as $z^4+6Hz^2-4Gz+(a_0^2I-3H^2)=0$, then its Euler's cubic will be the same as that of the given biquadratic and hence both will have the same reducing cubic.

§ 5. Relations between the roots of the biquadratic and Euler's cubic. (Agra 51)

We have taken $z=\sqrt{p}+\sqrt{q}+\sqrt{r}$ ... ...(1)

If each of the three radicals have their double sign, then we shall get eight values of z by the above assumption, but there is a condition that

$$\sqrt{p}.\sqrt{q}.\sqrt{r}=-\frac{G}{2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots(2)$$

or

$$\sqrt{r}=\frac{-G}{2\sqrt{p}\sqrt{q}}$$

$$\therefore \quad z=\sqrt{p}+\sqrt{q}-\frac{G}{2\sqrt{p}\sqrt{q}}.$$

Now giving $\sqrt{p}$, $\sqrt{q}$ the double signs, we get the four values of z.

The signs to be affixed to $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ should be such as to satisfy the limitation imposed by (2), *i.e.* if G be $-$ive, *i. e.* $-\frac{G}{2}$ $+$ive, we should affix such signs as to make $\sqrt{p}\sqrt{q}\sqrt{r}$ also $+$ive. In this case we shall choose all of them to be $+$ive or two $-$ive and one $+$ive.

If z_1, z_2, z_3, z_4 be the values of z and α, β, γ, δ be the roots of the given biquadratic, then

$G-$*ive.*

$$\left.\begin{array}{l} z_1=a_0\alpha+a_1=\sqrt{p}-\sqrt{q}-\sqrt{r} \\ z_2=a_0\beta+a_1=-\sqrt{p}+\sqrt{q}-\sqrt{r} \\ z_3=a_0\gamma+a_1=-\sqrt{p}-\sqrt{q}+\sqrt{r} \\ z_4=a_0\delta+a_1=\sqrt{p}+\sqrt{q}+\sqrt{r} \end{array}\right. \left\{\begin{array}{ll} \sqrt{p} & +\text{ive} \\ \sqrt{q} & +\text{ive} \\ \sqrt{r} & +\text{ive} \\ \text{all} & +\text{ive} \end{array}\right\} \ldots\ \ldots\text{(A)}$$

$$\therefore\quad a_0(\alpha+\beta)+2a_1=-2\sqrt{r}$$

$$a_0(\gamma+\delta)+2a_1=2\sqrt{r}.$$

$$\therefore\quad a_0(\alpha+\beta-\gamma-\delta)=-4\sqrt{r}$$

or

$$r=\frac{a_0^2}{16}(\alpha+\beta-\gamma-\delta)^2.$$

$$\text{Similarly,}\quad \left.\begin{array}{l} q=\dfrac{a_0^2}{16}(\gamma+\alpha-\beta-\delta)^2 \\ p=\dfrac{a_0^2}{16}(\beta+\gamma-\alpha-\delta)^2. \end{array}\right\}\ \ldots\ \ldots\ \ldots\text{(B)}$$

(Remember)

Note—In relations. (B), δ is always with minus sign and α, β, γ are $-$ive respectively in the values p, q and r.

Note—In case G be $+$ive *i.e.* $-\dfrac{G}{2}$ $-$ive, then we shall choose all $-$ive or two $+$ive and one $-$ive, *i.e.*

$$z_1=a_0\alpha+a_1=-\sqrt{p}+\sqrt{q}+\sqrt{r}\ \text{etc.}\quad \sqrt{p}\ -\text{ive}.$$

Q. 1. *Prove that if the biquadratic has two distinct pairs of equal roots, then $a_0^2I=12H^2$ and $a_0^3J=8H^3$.* **(Pb. 45, 60)**

Suppose $\alpha=\beta$ and $\gamma=\delta$, then from relations (B) both p and q are zero *i.e.* two of the roots of the Euler's cubic

$$t^3+3Ht^2+\left(3H^2-\frac{a_0^2I}{4}\right)t-\frac{G^2}{4}=0$$

vanish, and hence

$$G=0\ \text{and}\ 3H^2=\frac{a_0^2I}{4}\ \textit{i.e.}\ a_0^2I=12H^2.$$

Again we know that $G^2+4H^3=a_0^2(HI-a_0J)$.

Putting $G=0$ and $a_0^2I=12H^2$, we get $a_0^3J=8H^3$.

Q. 2. *Prove the following :—*

(i) When the roots of the biquadratic are all real, the roots of Euler's cubic are all real and +ive.

(ii) When the biquadratic has all its roots imaginary, the roots of Euler's cubic are all real, two being negative and one positive.

(iii) When the biquadratic has two roots real and two imaginary, then Euler's cubic has two imaginary and one real +ive root.

Make use of relations (B) and take imaginary conjugate roots as $a \pm ib$ etc.

Q. 3. *Find the value of*

$$a_0^3\,(\beta+\gamma-\alpha-\delta)\,(\gamma+\alpha-\beta-\delta)\,(\alpha+\beta-\gamma-\delta)$$

in terms of the coefficients of the biquadratic. **(Agra 35)**

From (B), we have $a_0(\alpha+\beta-\gamma-\delta)=-4\sqrt{r}$ etc.

Hence the given function$=(-4\sqrt{r})\,(-4\sqrt{p})\,(-4\sqrt{q})$

$$-64\sqrt{p}.\sqrt{q}.\sqrt{r}=-64\left(\frac{-G}{2}\right) \text{ from (2)}$$

$$=32G=32\,[a_0^2a_3-3a_0a_1a_2+2a_1^3].$$

Q. 4. *Prove that if two roots of Euler's cubic vanish, then the biquadratic* $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ *has two pairs of equal roots given by* $\dfrac{-a_1 \pm \sqrt{(-3H)}}{a_0}$.

If two roots of Euler's cubic vanish, then from Q. 1,

$$G=0 \text{ and } a_0^2I=12H^2 \quad \ldots \quad \ldots \quad \ldots(1)$$

Also if $z=a_0x+a_1$ then the given biquadratic reduces

to $$z^4+6Hz^2+4Gz+a_0^2I-3H^2=0,$$

or $$z^4+6Hz^2+9H^2=0 \quad \text{from (1)}.$$

$$\therefore \quad (z^2+3H)^2=0 \quad \text{or} \quad z=\pm\sqrt{(-3H)}$$

or $$a_0x+a_1=\pm\sqrt{(-3H)}, \quad \therefore \quad x=\frac{-a_1\pm\sqrt{(-3H)}}{a_0}.$$

Note. $a_0^3f(x)=[(a_0x+a_1)^2+3H]^2$ by putting the value of z in $(z^2+3H)^2$.

Q. 5. *When a biquadratic has two equal roots, prove that Euler's cubic has two equal roots whose common value is* $\frac{3aJ-2HI}{2I}$ *and hence show that the remaining two roots of the biquadratic in this case are real, equal or imaginary according as* $3aJ-2HI$ *is* $+$*ive, zero or* $-$*ive.*

(Agra 1934, 38, 42)

Let $\alpha=\beta$; then from relations (B), we get

$$p=\frac{a^2}{16}(\gamma-\delta)^2=q.$$

Hence the Euler's cubic,

i.e. $$f(t)=t^3+3Ht^2+\left(3H^2-\frac{a^2I}{4}\right)t-\frac{G^2}{4}=0$$

has two equal roots.

Now we know that if $f(t)$ has two equal roots, then $f(t)$ and $f'(t)$ must have H. C. F. of 1st degree which when equated to zero will give the required root. In the process of finding the H. C. F. the first degree quotient is

$$\frac{-a^2I}{6}t-\frac{G^2}{4}-H^3+\frac{a^2IH}{12}$$

which should be the H. C. F. and when equated to zero gives the required equal root.

$$\therefore \quad t=\frac{\left\{\frac{a^2IH}{12}-\frac{G^2+4H^3}{4}\right\}}{\frac{a^2I}{6}}.$$

Putting $G^2+4H^3=a_0^2\,(HI-a_0J)$, we get

$$t=\frac{3aJ-2HI}{2I}=p \quad \text{or} \quad q=\frac{a_0^2}{16}(\gamma-\delta)^2.$$

Now if $3aJ-2HI$ is +ive, then $(\gamma-\delta)^2$ is +ive and as such both γ and δ must be real. If $3aJ-2HI=0$, then clearly $\gamma=\delta$ and if $3aJ-2HI$ be −ive, then $(\gamma-\delta)^2$ being −ive both γ and δ should be conjugate complex.

§ 6. Relation between the roots of the biquadratic and reducing cubic $4a_0^3\theta^3-Ia_0\theta+J=0$.

From relation (A) of § 5, we have

$$a_0(\alpha-\beta)=2(\sqrt{p}-\sqrt{q})$$

and

$$a_0(\gamma-\delta)=-2(\sqrt{p}+\sqrt{q}).$$

$$\therefore\quad -a_0^2\,(\alpha-\beta)\,(\gamma-\delta)=4\,(p-q)=4a_0^2\,(\theta_1-\theta_2).$$

$$\because\quad t+H=a_0^2\theta,$$

$$\therefore\quad \left.\begin{aligned} 4\,(p-q)&=4a_0^2\,(\theta_1-\theta_2)=-a_0^2\,(\alpha-\beta)\,(\gamma-\delta)\\ 4\,(q-r)&=4a_0^2\,(\theta_2-\theta_3)=-a_0^2\,(\beta-\gamma)\,(\alpha-\delta)\\ 4\,(r-p)&=4a_0^2\,(\theta_3-\theta_1)=-a_0^2\,(\gamma-\alpha)\,(\beta-\delta)\end{aligned}\right\}\ldots\ldots(\mathrm{C})$$

Remember the above relations.

Note. The first factor in the relation of $p-q$ is $(\alpha-\beta)$ and the 2nd is $(\gamma-\delta)$. In the 2nd factor δ is always −ive.

Subtracting 1st relation of (C) from the 3rd, we get

$$4a_0^2\,[\theta_3-\theta_1-\theta_1+\theta_2]=-a_0^2\,[(\gamma-\alpha)\,(\beta-\delta)-(\alpha-\beta)\,(\gamma-\delta)].$$

Put $\theta_1+\theta_2+\theta_3=0$; $\therefore\ \theta_2+\theta_3=-\theta_1$.

$$\left.\begin{aligned} \therefore\quad 12\theta_1&=(\gamma-\alpha)\,(\beta-\delta)-(\alpha-\beta)\,(\gamma-\delta).\\ \text{Similarly}\quad 12\theta_2&=(\alpha-\beta)\,(\gamma-\delta)-(\beta-\gamma)\,(\alpha-\delta)\\ \text{and}\quad 12\theta_3&=(\beta-\gamma)\,(\alpha-\delta)-(\gamma-\alpha)\,(\beta-\delta)\end{aligned}\right\}\ldots\ldots(\mathrm{D})$$

Remember the above relations.

Ex. 1. *Find the values of the following symmetric functions of the roots of the biquadratic in terms of the coefficients :—*

(1) $\{(\beta-\gamma)^2(\alpha-\delta)^2+(\gamma-\alpha)^2(\beta-\delta)^2+(\alpha-\beta)^2(\gamma-\delta)^2\}.$

(Agra 41)

(2) $$\begin{aligned}&[(\gamma-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta)]\\ \times&[(\alpha-\beta)(\gamma-\delta)-(\beta-\gamma)(\alpha-\delta)]\\ \times&[(\beta-\gamma)(\alpha-\delta)-(\gamma-\alpha)(\beta-\delta)].\end{aligned}$$

(3) $(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2(\alpha-\delta)^2(\beta-\delta)^2(\gamma-\delta)^2,$

i.e. product of the squares of the difference of the roots of the biquadratic.

(1) From relations (C) the given symmetric function is

$$16\,[(\theta_1-\theta_2)^2+(\theta_2-\theta_3)^2+(\theta_3-\theta_1)^2]\ldots\quad\ldots\quad\ldots(1)$$

where θ_1, θ_2, θ_3 are the roots of the cubic

$$4a_0^3\theta^3-Ia_0\theta+J=0.$$

We have to find the value of

$$16\,[2\Sigma\theta_1^2-2\Sigma\theta_1\theta_2]=32\,[(\Sigma\theta_1)^2-2\Sigma\theta_1\theta_2-\Sigma\theta_1\theta_2]$$

$$=-96\left[\frac{-Ia_0}{4a_0^3}\right]\quad\because\quad\Sigma\theta_1=0\ \text{and}\ \Sigma\theta_1\theta_2=\frac{-Ia_0}{4a_0^3}$$

$$=\frac{24}{a_0^2}\,[a_0a_4-4a_1a_3+3a_2^2].$$

(2) The given symmetric function from relation (*D*), is

$$12^3\theta_1\theta_2\theta_3=12\times12\times12\times\left[\frac{-J}{4a_0^3}\right]$$

$$=\frac{-432}{a_0^3}\,[a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3]$$

(3) The given symmetric function from relation (*C*) is $16^3(\theta_1-\theta_2)^2(\theta_2-\theta_3)^2(\theta_3-\theta_1)^2$ where θ_1, θ_2, θ_3 are the roots

of $\theta^3-\frac{I\theta}{4a_0^2}+\frac{J}{4a_0^3}=0$ which is of the form.

(See § 13 P. 77)

$x^3+qx+r=0$ whose squared difference equation is $y^3+6qy^2+9q^2y+(4q^3+27r^2)=0$. The product of the roots $=-(4q^3+27r^2)$. Therefore the product of the roots of the squared difference of reducing cubic is

$$-\left(27.\frac{J^2}{16a_0^6}-4.\frac{I^3}{64a_0^6}\right)=\frac{1}{16a_0^6}(I^3-27J^2).$$

Hence given symmetric function is

$$\frac{256}{a_0^6}(I^3-27J^2).$$

(Agra 47)

Discriminant :—The expression I^3-27J^2 is called discriminant of the biquadratic and is denoted by Δ. $a_0^6(\alpha-\beta)^2(\alpha-\gamma)^2(\alpha-\delta)^2(\beta-\gamma)^2(\beta-\delta)^2(\gamma-\delta)^2=256\Delta$. The vanishing of the discriminant Δ is the necessary and sufficient condition for the equation to have equal roots.

Note. We know that from a given equation to obtain another equation whose roots are the roots of the given equation with sign changed, we have to change the sign of every alternate term beginning from the 2nd. Also we know that $(\alpha-\beta)^2=[(-\alpha)-(-\beta)]^2$ etc.

Also discriminant of a cubic or biquadratic when multiplied by a constant gives the product of squared difference of the roots of a given equation and hence we conclude that the discriminant of a given equation remains unchanged if we change the sign of every alternate term of the given equation beginning from the 2nd (the equation being complete).

Ex. 2. *If the roots* $\alpha, \beta, \gamma, \delta$ *of the biquadratic*

$$ax^4+4bx^3+6cx^2+4dx+e=0$$

be so related that $\alpha-\delta$, $\beta-\delta$ *and* $\gamma-\delta$ *be in H. P., prove that*

$$ace+2bcd-ad^2-b^2e-c^3=0. \qquad \textbf{(Delhi Hon's 52)}$$

Since $\quad \alpha-\delta, \beta-\delta, \gamma-\delta$ are in H. P.

or $$\frac{1}{\alpha-\delta}, \frac{1}{\beta-\delta}, \frac{1}{\gamma-\delta} \text{ are in A. P.}$$

or $$\frac{1}{\beta-\delta}-\frac{1}{\alpha-\delta}=\frac{1}{\gamma-\delta}-\frac{1}{\beta-\delta}$$

ar $$\frac{(\alpha-\beta)}{(\beta-\delta)(\alpha-\delta)}=\frac{(\beta-\gamma)}{(\gamma-\delta)(\beta-\delta)}$$

or $$(\alpha-\beta)(\gamma-\delta)-(\beta-\gamma)(\alpha-\delta)=0$$

i. e. $\quad 12\theta_2=0$ or $\theta_2=0$ [from relation (D) P. 137]

Hence one root of the equation $4a_0^3\theta^3-Ia\theta+J=0$ is zero and consequently $J=0$ or $ace+2bcd-ad^2-b^2e-c^3=0.$

Resolution of the biquadratic into quadratic factors.

§ 7. Descarte's Method. (Delhi Hon's 48 ; Agra 36, 40, 47, 51, 54, 55 ; Pb. 35, 41, 54, 55, 60)

Let the equation be

$$f(x)\equiv a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$$

and it be reduced to the form

$$f(z)\equiv z^4+6Hz^2+4Gz+a_0^2I-3H^2=0. \qquad \textbf{(Agra 41)}$$

where $z=a_0x+a_1$. If z_1, z_2, z_3, z_4 be the roots of the equation $f(z)=0$, then $z_1+z_2+z_3+z_4=0$

or $$(z_1+z_2)=-(z_3+z_4)=p \text{ say}$$

and let $z_1z_2=q$ and $z_3z_4=q'$

$$\therefore \quad f(z)\equiv(z^2-pz+q)(z^2+pz+q').$$

Comparing the coefficients of like powers of z, we get

$$\left.\begin{array}{ll}\text{(i)} & q+q'-p^2=6H \\ \text{(ii)} & p(q-q')=4G \\ \text{(iii)} & qq'=a_0^2I-3H^2\end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(A)}$$

Now $$(q-q')^2=(q+q')^2-4qq'$$

or $$\frac{16G^2}{p^2}=(p^2+6H)^2-4\,(a_0^2I-3H^2)$$

or $$p^2\,(p^4+12Hp^2+36H^2)-4p^2\,(a_0^2I-3H^2)-16G^2=0$$

or $$p^6+12Hp^4+4p^2\,(12H^2-a_0^2I)-16G^2=0.$$

Above equation being a cubic in p^3 can be solved generally by trial method and having found p^2 we can find the values of q and q' from (A) and then the quadratic factors of $f(z)$ are known. In this way we can find the roots of $f(z)=0$ and consequently of $f(x)=0$.

Reducing Cubic :—The cubic in p^2 can be put as

$$(p^6+12Hp^4+48p^2H^2+64H^3)-4a_0^2I\,(p^2+4H)$$
$$-16G^2-64H^3+16a_0^2IH$$

or $$(p^2+4H)^3-4a_0^2I\,(p^2+4H)-16\,(-a_0^3J)$$
$$\because\quad G^2+4H^3=a_0^2HI-a_0^3J.$$

Now putting $p^2+4H=4a_0^2\theta$, we get $4a_0^3\theta^3-Ia_0\theta+J=0$ which is the same reducing cubic as found in § 4.

Note :—In order to find the value of p^2 by trial, try those numbers which are whole squares *i. e.* 1, 4, 16 etc. and see if they satisfy the cubic.

Solve by Descarte's method the following equations :

Ex. 1. $x^4-3x^2-42x-40=0.$ **(Agra 54)**

Let $f(x)\equiv(x^2+px+q)\,(x^2-px+q')$.

Comparing the coefficients, we get

$$q+q'-p^2=-3 \quad \text{or} \quad q+q'=p^2-3$$

$$p\,(q'-q)=-42 \quad \text{or} \quad q'-q=-\frac{42}{p}$$

and $$qq'=-40$$

$$(q-q')^2=(q+q')^2-4qq'$$

or
$$\frac{42^2}{p^2}=(p^2-3)^2+160$$

or
$$t^3-6t^2+169t-1764=0,$$

where $t=p^2$. Clearly $t=9$ satisfies the above cubic.

$$\therefore \quad p^2=9 \quad \text{or} \quad p=3 \quad \text{or} \quad -3.$$

Choosing $p=3$, we get $q+q'=6$ and $q'-q=-14$.

$\therefore$ $q'=-4$ and $q=10$ and these values satisfy the relations $qq'=-40$.

Hence $f(x)\equiv(x^2+3x+10)(x^2-3x-4)=0$.

$$\therefore \quad x=4,\ -1,\ \frac{-3\pm i\sqrt{31}}{2}.$$

Ex. 2. $x^4-5x^2-6x-5=0$. **(Pb. 35, 48; Delhi Hon's 51)**

Proceeding as above, we get $t^3-10t^2+45t-36=0$, where $t=p^2=1$ etc. **Ans.** $\dfrac{-1\pm i\sqrt{3}}{2}$, $\dfrac{1\pm\sqrt{(21)}}{2}$.

Ex. 3. $x^4-6x^3+3x^2+22x-6=0$. **(Agra 40, 47)**

In order to remove the 2nd term, we shall have to diminish the roots by $\frac{3}{2}$ which will be difficult. Hence we first multiply the roots by 2 and get

$$y^4-12y^3+12y^2+176y-96=0.$$

Now diminish the roots by 3 and we get

$$z^4-42z^2+32z+297=0$$

where
$$z=y-3=2x-3, \text{ etc.}$$

In order to solve $f(z)$ proceed as above and we get

$t^3-84t^2+576t-1024=0$, where $t=p^2=4$ etc.

$\therefore$ $(x^2-2x-6)(x^2-4x+1)=0$ or $x=1\pm\sqrt{7},\ 2\pm\sqrt{3}$.

Ex. 4. $x^4-6x^3-9x^2+66x-22=0$. **(Agra 36)**

$t^3-180t^2+20\times272t-96\times96=0$, where $t=p^2=36$.

$$\therefore \quad (x^2-4)(x^2-6x+2)=0.$$

Ex. 5. $x^4-8x^2-24x-7=0.$ **(Pb. 31, 60)**

Hint. $t^3-16t^2+36t-576=0$, $t=16=p^2$ etc.

Ans. $-2\pm i\sqrt{3}$, $2\pm\sqrt{3}$.

Ex. 6. $x^4-10x^2-20x-16=0.$ **(Pb. 41)**

Hint. $t^3-20t^2+164t-400=0$, where $t=4=p^2$ etc.

Ans. $x=4,\ -2,\ -1\pm i.$

Ex. 7. $x^4-2x^2+8x-3.$ **(Pb. 43)**

Ans. $p^2=4$, $1\pm i\sqrt{2}$, $-1\pm\sqrt{2}$.

Ex. 8. (a) *Solve* $x^4-8x^3-12x^2+60x+63=0.$

In order to remove the 2nd term, first diminish the roots by 2 and the resulting equation is

$$z^6-36z^2-52z+87=0, \text{ where } z=x-2.$$

Now proceeding as usual, t cubic is

$$t^3-72t^2+12\times 79t-52\times 52=0, \text{ where } t=p^2=4.$$

$$\therefore\ f(z)=(z^2+2z-3)(z^2-2z-29).$$

$$\therefore\ z=-3, 1,\ 1\pm\sqrt{30}.$$

Hence $x=z+2=-1,\ 3,\ 3\pm\sqrt{30}$.

(b) $x^4+8x^3+9x^2-8x-10=0.$

Ans. ± 1, $-4\pm\sqrt{6}$.

Ex. 9. (a) $x^4-12x-5=0.$ **(Pb. 55)**

Hint. $t^3+20t-144=0.$ $\therefore\ p^2=t=4.$ $\therefore\ p=2$ etc.

Ans. $-1\pm 2i$, $1\pm\sqrt{2}$.

(b) *Resolve into quadratic factors*

$$x^4+12x+3=0.$$

Proceeding as usual, t cubic is

$$t^3-12t-144=0, \text{ where } t=p^2=6. \quad \therefore\ p=\sqrt{6}.$$

$$\therefore\ f(x)=(x^2+\sqrt{6}x+3-\sqrt{6})(x^2-\sqrt{6}x+3+\sqrt{6}).$$

Ex. 10. $x^4-3x^2-6x-2=0.$ **(Agra 55)**

$$t^3-6t^2+17t-36=0, \text{ where } t=p^2=4 \text{ etc.}$$

$$\therefore\ f(x)=(x^2+2x+2)(x^2-2x-1)=0.$$

$$\therefore\ x=-1\pm i,\ 1\pm\sqrt{2}.$$

§ 8. Ferrari's Method.

(Agra 44, 46, 48, 50, 52, 57, 60; Pb. 33, 39)

Let $f(x)=ax^4+4bx^3+6cx^2+4dx+e=0$.

Let us suppose that

$$f(x)\equiv\frac{1}{a}\{(ax^2+2bx+c+2a\theta)^2-(2Mx+N)^2\}.$$

Comparing the coefficients of like powers of x, we get

$$M^2=b^2-ac+a^2\theta \quad \ldots \quad \ldots \quad \ldots(1)$$

$$MN=bc-ad+2ab\theta \quad \ldots \quad \ldots \quad \ldots(2)$$

$$N^2=(c+2a\theta)^2-ae \quad \ldots \quad \ldots \quad \ldots(3)$$

Now $\qquad M^2N^2=(MN)^2$.

$$\therefore \quad (b^2-ac+a^2\theta)\{(c+2a\theta)^2-ae\}=(bc-ad+2ab\theta)^2.$$

The above equation on simplification reduces to

$$4a^3\theta^3-(ce-4bd+3c^2)\,a\theta+ace-2bcd-ad^2-eb^2-c^3$$

or

$$4a^3\theta^3-Ia\theta+J=0 \quad \ldots \quad \ldots \quad \ldots(4)$$

which is the same reducing cubic as before. From this equation, we can find the values of θ and hence the values of M and N. In this way we are able to express the given biquadratic as difference of two squares and we obtain the two quadratics

$$ax^2+2(b-M)x+c+2a\theta-N=0 \quad \ldots \quad \ldots(5)$$

and

$$ax^2+2(b+M)x+c+2a\theta+N=0 \quad \ldots \quad \ldots(6)$$

Now θ will have three values say θ_1, θ_2, θ_3 and corresponding values of M be M_1, M_2, M_3 and those of N be N_1, N_2, N_3.

Let the roots of (5) be β, γ and those of (6) be α, δ when M, N, θ receive the values M_1, N_1 and θ_1 respectively.

$$\therefore \quad \beta+\gamma=-\frac{2}{a}(b-M_1) \text{ and } \beta\gamma=\frac{c+2a\theta_1-N_1}{a}$$

$$\alpha+\delta=-\frac{2}{a}(b+M_1) \text{ and } \alpha\delta=\frac{c+2a\theta_1+N_1}{a}.$$

$$\left.\begin{aligned} \therefore\quad \beta+\gamma-\alpha-\delta&=\frac{4}{a}M_1.\\ \text{Similarly}\quad \gamma+\alpha-\beta-\delta&=\frac{4}{a}M_2\\ \text{and}\quad \alpha+\beta-\gamma-\delta&=\frac{4}{a}M_3 \end{aligned}\right\}\quad \ldots\quad \ldots\quad \ldots\text{(A)}$$

$$\left.\begin{aligned} \beta\gamma+\alpha\delta&=\frac{2c}{a}+4\theta_1\\ \text{and}\quad \gamma\alpha+\beta\delta&=\frac{2c}{a}+4\theta_2\\ \text{and}\quad \alpha\beta+\gamma\delta&=\frac{2c}{a}+4\theta_3 \end{aligned}\right\}\quad \ldots\quad \ldots\quad \ldots\text{(B)}$$

Also from (A), we get

$$\begin{aligned} \frac{4}{a}(M_2+M_3-M_1)&=(2\alpha-2\delta-\beta-\gamma+\alpha+\delta)\\ &=(3\alpha-\beta-\gamma-\delta)=(4\alpha-\Sigma\alpha)\\ &=\left(4\alpha+\frac{4b}{a}\right)=\frac{4}{a}(a\alpha+b). \end{aligned}$$

Also from (A), we get

$$\left.\begin{aligned} a\alpha+b&=-M_1+M_2+M_3\\ a\beta+b&=M_1-M_2+M_3\\ a\gamma+b&=M_1+M_2-M_3\\ \text{and}\quad a\delta+b&=-M_1-M_2-M_3 \end{aligned}\right\}\ldots\text{(C)}\qquad \left.\begin{aligned} \beta\gamma-\alpha\delta&=\frac{-2N_1}{a}\\ \gamma\alpha-\beta\delta&=\frac{-2N_2}{a}\\ \alpha\beta-\gamma\delta&=\frac{-2N_3}{a} \end{aligned}\right\}.\text{(D)}$$

Ex. 1. *If* α, β, γ, δ *be the roots of the biquadratic* $ax^4+4bx^3+6cx^2+4dx+e=0$. *form the equation whose roots are.*

(*i*) $\beta\gamma+\alpha\delta,\ \gamma\alpha+\beta\delta,\ \alpha\beta+\gamma\delta.$ **[See Ex. 7 (a) P. 93]**

(*ii*) $\dfrac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta},\ \dfrac{\gamma\alpha-\beta\delta}{\gamma+\alpha-\beta-\delta},\ \dfrac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta}.$

[See Ex. 7 P. 74] (Agra 46)

(iii) $\frac{1}{8}(\beta\gamma-\alpha\delta)(\beta+\gamma-\alpha-\delta)$, $\frac{1}{8}(\gamma\alpha-\beta\delta)(\gamma+\alpha-\beta-\delta)$,

$\frac{1}{8}(\alpha\beta-\gamma\delta)(\alpha+\beta-\gamma-\delta)$.

From relation (1), $z=\beta\gamma+\alpha\delta=\frac{2c}{a}+4\theta_1$

$\therefore\quad \theta_1=\frac{(az-2c)}{4a}$ where θ_1 is a root of $4a^3\theta^3-Ia\theta+J=0$.

Hence the required equation is

$$(ax-2c)^3-4I(ax-2c)+16J=0.$$

(ii) $z=\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta}=-\frac{1}{2}\frac{N_1}{M_1}$ from (A) and (D)

or $z=\frac{-1}{2}\frac{bc-ad+2ab\theta_1}{b^2-ac+a^2\theta_1}$ from (1) and (2), § 8

or $az+b=-\frac{1}{2}\left(\frac{abc-a^2d+2a^2b\theta_1-2b^3+2abc-2a^2b\theta_1}{b^2-ac+a^2\theta_1}\right)$

or $az+b=\frac{1}{2}\left(\frac{-3abc+a^2d+2b^3}{a^2\theta_1-H}\right)$

or $az+b=\frac{1}{2}\frac{-G}{a^2\theta_1-H}$.

Now eliminating θ_1 between above and $4a^3\theta_1^3-Ia\theta_1+J=0$, we get the required equation.

(iii) $z=\frac{1}{8}(\beta\gamma-\alpha\delta)(\beta+\gamma-\alpha-\delta)$

$=\frac{1}{8}.\left(\frac{-2N_1}{a}\right)\left(\frac{4M_1}{a}\right)=\frac{-M_1N_1}{a^2}$ from (A) and (D)

or $-a^2z=bc-ad+2ab\theta_1$

or $\theta_1=\left(\frac{a^2z+bc-ad}{2ab}\right)$,

where θ_1 satisfies the equation $(4a^3\theta^3-Ia\theta+J)=0$.

Hence the required equation is

$$(a^2z+bc-ad)^3-b^2I(a^2z+bc-ad)+2b^3J=0.$$

Ex. 2. *Show that* $M_1M_2M_3=\frac{G}{2}$.

We have from relation (A) of § 8.

$$M_1M_2M_3=\frac{a^3}{64}(\beta+\gamma-\alpha-\delta)(\gamma+\alpha-\beta-\delta)(\alpha+\beta-\gamma-\delta)$$

$$=\frac{1}{64}.32G=\frac{G}{2}.$$ **(See Ex. 3, P. 134)**

Alternative—Let α, β, γ, δ be the roots of

$$ax^4+4bx^3+6cx^2+4dx+e=0.$$

If its roots be diminished by $h=-\frac{b}{a}$, then we get

$$y^4+\frac{6Hy^2}{a^2}+\frac{4Gy}{a^3}+\frac{(a^2I-3H^2)}{a^4}=0$$

and its roots $\alpha-h$, $\beta-h$, $\gamma-h$, $\delta-h$ be α', β', γ', δ', then

$$\beta+\gamma-\alpha-\delta=\beta'+\gamma'-\alpha'-\delta'=-2(\alpha'+\delta').$$

$$\because\ \alpha'+\beta'+\gamma'+\delta'=0, \quad \therefore\ \beta'+\gamma'=-(\alpha'+\delta').$$

$$\therefore\quad M_1M_2M_3=\frac{a^3}{64}\{-8(\alpha'+\delta')(\beta'+\delta')(\gamma'+\delta')\}$$

[from (A) P. 145)

$$=-\frac{a^3}{8}\{\delta'^3+\delta'^2(\alpha'+\beta'+\gamma')+\delta'(\alpha'\beta'+\beta'\gamma'+\gamma'\alpha')+\alpha'\beta'\gamma'\}$$

$$=-\frac{a^3}{8}\{\delta'^2(\alpha'+\beta'+\gamma'+\delta')+\Sigma\alpha'\beta'\gamma'\}$$

$$=-\frac{a^3}{8}(\Sigma\alpha'\beta'\gamma'+\delta'^2\Sigma\alpha')$$

$$=-\frac{a^3}{8}\times-\frac{4G}{a^3}=\frac{G}{2}.$$ Hence proved

Note. Solution of biquadratic equation by Ferrari's method for numerical equation.

The following example will illustrate the above method in a simple form.

Ex. 3. *Solve the equation*

$$x^4+2x^3-7x^2-8x+12=0.$$ **(Agra 52)**

$$x^4+2x^3=7x^2+8x-12.$$

Adding x^2 to both sides,

$$(x^2+x)^2=8x^2+8x-12.$$

$$\therefore \quad (x^2+x+\lambda)^2=8x^2+8x-12+\lambda^2+2\lambda\,(x^2+x)$$

$$=x^2\,(8+2\lambda)+x\,(8+2\lambda)+\lambda^2-12.$$

Now apply the condition that R.H.S. be a perfect square *i.e.* $B^2-4AC=0$ and we shall get a cubic in λ from which λ can be found. The solution of cubic in λ may not be easy and as such we try the following method. If R.H.S. is to be a perfect square, then the coefficient of x^2 as well as constant term both should be perfect squares. We put the coefficient of x^2 equal to 1, 4, 9, 16 or $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$ etc., and see that the value of λ thus obtained makes the constant term also a perfect square.

In this case $8+2\lambda=1$ or $\lambda=-\frac{7}{2}$ and this value of λ makes λ^2-12 equal to $\frac{1}{4}$ which is a perfect square.

$$\therefore \quad \text{R.H.S.}=x^2+x+\tfrac{1}{4}=(x+\tfrac{1}{2})^2.$$

$$\therefore \quad f(x)=(x^2+x-\tfrac{7}{2})^2-(x+\tfrac{1}{2})^2=0$$

or

$$(x^2+2x-3)\,(x^2-4)=0$$

$$\therefore \quad x=\pm 2,\ -3,\ 1.$$ **Ans.**

Ex. 4. (a) $2x^4+6x^3-3x^2+2=0.$

(Delhi Hons. 50, Pb. 33, Agra 60)

Multiplying the roots by (2), we get

$$f(z)=z^4+6z^3-6z^2+16=0, \qquad \text{where } z=2x$$

or

$$(z^2+3z)^2=15z^2-16$$

or

$$(z^2+3z+\lambda)^2=15z^2-16+\lambda^2+2\lambda\,(z^2+3z).$$

R.H.S. is $z^2\,(15+2\lambda)+6\lambda z+(\lambda^2-16)=0.$

Let $15+2\lambda=25$, $\therefore$ $\lambda=5$ and R. H. S. is $(5z+3)^2$.

$$\therefore \quad f(z)=(z^2+3z+5)^2-(5z+3)^2=0$$

or

$$(z^2+8z+8)\,(z^2-2z+2)=0,$$

$$x=\frac{z}{2}=-2\pm\sqrt{2},\ \frac{1\pm i}{2}.$$

(b) *Resolve into quadratic factors*

$x^4+3x^3+x^2-2.$ **(Agra 48)**

$$(x^2+\tfrac{3}{2}x+\lambda)^2=(2\lambda+\tfrac{5}{4})\,x^2+3\lambda x+\lambda^2+2,$$

where $2\lambda+\frac{5}{4}=\frac{1}{4}$ or $\lambda=-\frac{1}{2}$,

makes the R.H.S. a perfect square.

Hence $f(x)=(x^2+2x-2)\,(x^2+x+1)=0$.

Ex. 5. $x^4-10x^3+35x^2-50x+24=0.$ **(Pb. 39)**

$$(x^2-5x+\lambda)^2=(2\lambda-10)\,x^2+x\,(50-10\lambda)+\lambda^2-24,$$

where $2\lambda-10=4$ *i.e.* $\lambda=7$ etc. Roots are 1, 2, 3, 4.

Ex. 6. $x^4-8x^3-12x^2+60x+63=0.$ **(Agra 44)**

$$\lambda=-12,\ (x^2-6x-21)\,(x^2-2x-3)=0,$$

$$3\pm\sqrt{30},\ 3,\ -1,$$

Ex. 7. $x^4-2x^3-5x^2+10x-3=0.$ **(Agra 57)**

$$\lambda=-1,\ \frac{3\pm\sqrt{5}}{2},\ \frac{-1\pm\sqrt{13}}{2}.$$

Ex. 8. $z^4+4z^3+12z^2-8z+95=0.$

(Agra 40, 50; Pb. 47)

$$\lambda=12,\ (z^2+6z+19)\,(z^2-2z+5)=0.$$

Ex. 9. *If* $a\,(ax^4+4bx^3+6cx^2+4dx+e)$

$$\equiv(ax^2+2px+r)\,(ax^2+2qx+s) \text{ and } r+s=2\,(c-2\phi),$$

find the cubic equation giving ϕ. **(Delhi Hon's 50)**

Ex. 10. *Prove that* $u\equiv z^4+6Hz^2+4Gz+(a^2I-3H^2)$ *can be written as* $(z^2+3H+2k^2)\pm\left(2kz-\frac{G}{k}\right)$ *provided that* k^2+H *is a root of the equation* $4\phi^3-Ia^2\phi+a^3J=0$,

where $G^2+4H^3=a^2\,(HI-aJ)$. **(Delhi Hon's 49)**

CHAPTER VI

CHARACTER AND POSITION OF THE ROOTS OF AN EQUATION

§ 1. Descarte's Rule of Signs. (Pb. 31, 33, 50)

Positive Roots. *An equation $f(x)=0$ cannot have more positive roots than the number of changes of signs from + to − or from − to + in terms of its first member.*

Negative Roots. *An equation $f(x)=0$ cannot have more negative roots than the number of changes of sign in $f(-x)=0$.*

We shall simply verify the above statement and rigorous proof of the same is beyond the scope of this book.

Let the signs of a polynominal be

$$+ \quad + \quad - \quad + \quad - \quad + \quad - \quad -.$$

The given polynomial has five changes of sign. Now we shall multiply the given polynomial by a binomial $x-h$ corresponding to the +ive root h. The signs of this binomial are + −. We are concerned only with the signs and hence we multiply as below :—

$$\begin{array}{cccccccccl}
+ & + & - & + & - & + & - & - & & \text{5 changes of sign} \\
+ & - & & & & & & & & \\
\hline
+ & + & - & + & - & + & - & - & & \\
 & - & - & + & - & + & - & + & + & \\
\hline
+ & \pm & - & + & - & + & - & \pm & + &
\end{array}$$

The resulting polynomial has two ambiguous signs and we can write it in four different way as follows :—

A								*A*	6 changes	of sign
+	+	−	+	−	+	−	+	+	"	"
+	+	−	+	−	+			+	"	"
+	−	−	+	−	+			+	"	"
+	−	−	+	−	+	−	+	+	"	"

Thus we see that in all the four possible ways the resulting polynomial has six changes of signs *i. e.* one more than the number of changes of signs in the original polynomial. Hence we conclude that corresponding to the introduction of a positive root the resulting polynomial has one more change of sign. Now if $\phi(x)$ be the product of factors corresponding to $-$ve and complex roots, and $\alpha, \beta, \gamma \ldots$ be the $+$ve roots, then if $\phi(x)$ be multiplied by $(x-\alpha), (x-\beta), (x-\gamma) \ldots\ldots\ldots$ in succession, then each multiplication will introduce one more change of sign. Hence the number of positive roots cannot exceed the number of changes of sign in $f(x)=0$.

Negative roots. We know that $-$ive roots of $f(x)=0$ are positive roots of $f(-x)=0$ and as such the number of $-$ive roots of $f(x)=0$ cannot exceed the number of changes of signs of $f(-x)=0$.

Complex roots. If $f(x)=0$ be an equation of nth degree and if it be **complete**, then the number of changes of signs in $f(x)$ *i. e.* $+$ive roots and number of changes of signs in $f(-x)$ *i. e.* $-$ive roots is equal to nth degree of the equation and as such we cannot draw any definite conclusion regarding the existence of imaginary roots. In case the equation be **incomplete**, then the number of changes of signs in $f(x)$ *i. e.* $+$ive roots and the number of changes of signs in $f(-x)$ *i. e.* $-$ive roots is less than the degree n of the equation. If a and b be the number of changes of signs in $f(x)$ and $f(-x)$ respectively *i. e.* greatest number of +ive roots is a and that of −ive roots is b, then $n-(a+b)$ is the least number of imaginary roots. For example, consider the equation

$$f(x)=x^7-2x^5+7x^4+x^3-9=0.$$

The above equation has three changes of sign and as such it cannot have more than three +ive roots. Again $f(-x)=0$ *i. e.* $x^7-2x^5-7x^4+x^3+9=0$ has only two changes of sign and as such $f(x)=0$ cannot have more than two −ive roots. Thus the max. number of real roots is $3+2$ *i. e.* 5 and the degree of the equation being 7 we conclude that the equation must have at least two imaginary roots.

§ 2. Change of sign of f (x).

If two real numbers a and b be substituted for x in the polynomial f (x) and f (a) and f (b) are found to be of opposite signs, then at least one or an odd number of real roots of the equation f (x) = 0 lie between a and b. In case f (a) and f (b) be of the same sign, then either no real root or an even number of roots of f (x) = 0 lie between a and b. **(Agra Pb. 52)**

Case 1. f(a) and f(b) of opposite signs.

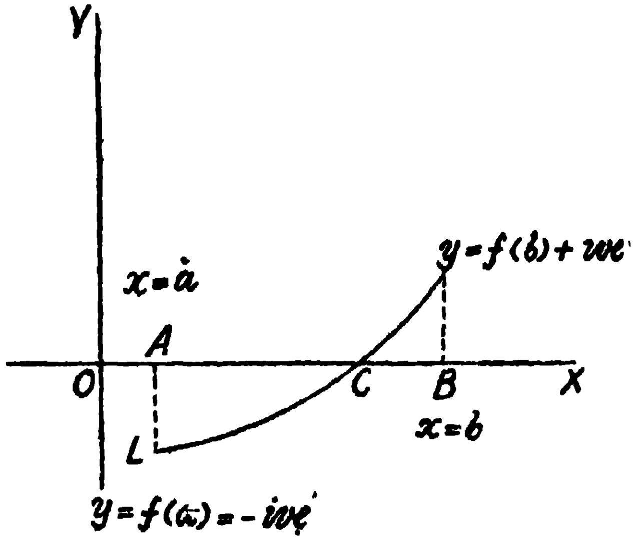


Fig 1.

Let $y=f(x)$ be a continuous function of x; it should assume all values between $f(a)$ and $f(b)$.

Now $f(a), f(b)$ are of opposite signs *i. e.* values of y corresponding to the values of x *i. e.* a and b are of opposite signs. From one side of x-axis to the other side of x-axis the curve $y=f(x)$ must cross the axis of x at least once as in fig 1 at C or an odd number of times as in fig 2 at C, D, and E and at all such points where the curve crosses the axis of x, $y=0$ *i. e.* $f(x)=0$ which means that $f(x)$ vanishes either at one or an odd number of times for values of x between a and b. Hence at least one or an odd number of roots of $f(x)$ lie between a and b.

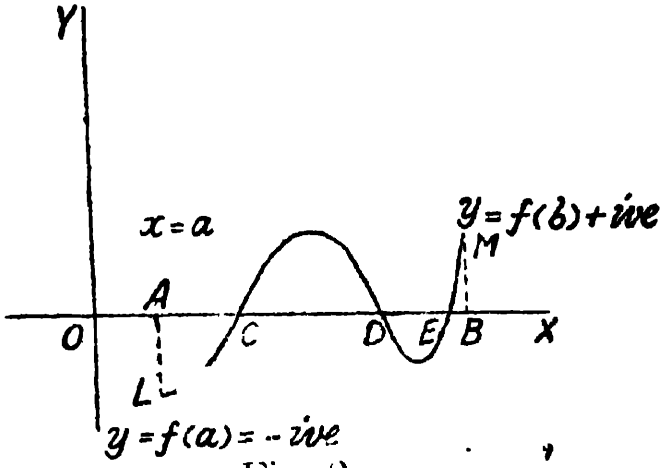


Fig. 2.

Case 2. f(a) and f(b) of same sign.

In this case $f(a)$ and $f(b)$ are of the same sign *i.e.* value of y corresponding to the values of x *i.e.* a and b are of the same sign which means that in passing from a point on one side of x-axis to another point on the same

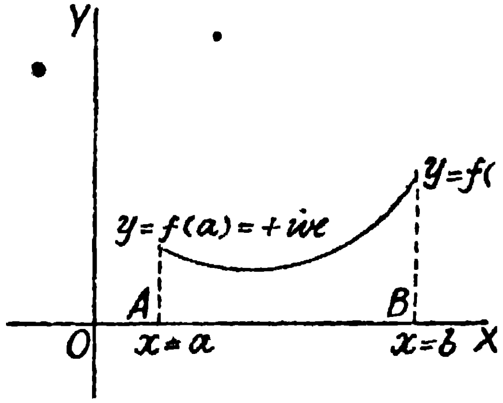


Fig. 3

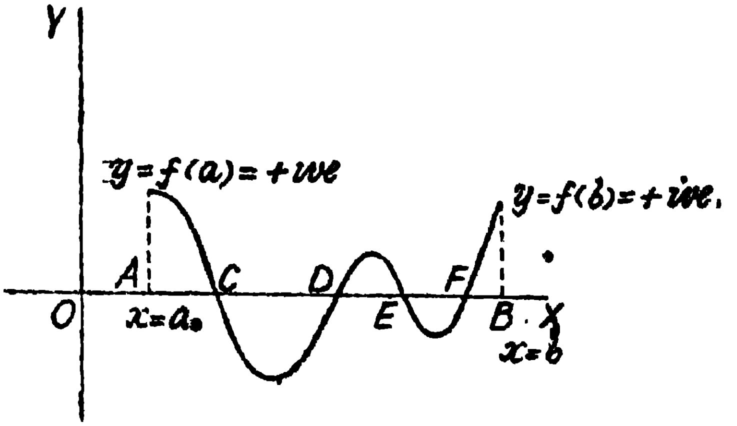


Fig. 4.

side either the curve $y=f(x)$ will not cross the x-axis as in Fig, 3 where $y=0$ *i.e.* $f(x)=0$, or, if it crosses, it must cross an even number of times as in Fig. 4. Hence we conclude that either $f(x)$ does not vanish for values of x between a and b, or, if it vanishes it must vanish for even number of times.

Deductions. 1. *Every equation of an odd degree has at least one real root whose sign is opposite to that of its last term, the coefficient of the first term being +ive.*

Let the equation be

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0 \quad (n \text{ odd}),$$

$$f(-\infty)=-\text{ive } (n \text{ odd}), \quad f(0)=p_n, \quad f(\infty)=+\text{ive}.$$

Case .I $\mathbf{p_n}$ is positive. In this case $f(-\infty)$ and $f(0)$ are of opposite sign and hence at least one real root must lie between $-\infty$ and 0 *i.e.* it must be negative opposite to the sign of p_n which is +ive,

Case II. $\mathbf{p_n}$ is negative. Here $f(0)=-$ive and $f(\infty)=$ +ive *i.e.* they are of opposite sign and hence at least one real root must lie between 0 and ∞. This root is clearly +ive *i.e.* of sign opposite to that of p_n.

2. *Every equation of an even degree whose last term is negative and the coefficient of the first term positive, has at least two real roots, one +ive and one −ive.*

Let the equation be

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=0 \qquad (n \text{ even})$$

$$f(-\infty)=+\text{ive}, f(0)=p_n \text{ i.e. } -\text{ive}, f(\infty)=+\text{ive}.$$

Hence $f(-\infty)$ and $f(0)$ are of opposite signs ; therefore at least one real root must lie between $-\infty$ and 0 and it is −ive. Again $f(0)$ and $f(\infty)$ are of opposite signs and hence at least ons real root must lie between 0 and ∞ and it is +ive. Thus the equation must have two real roots, one +ive and the other −ive.

3. *If an equation has only one change of sign, it must have one +ive root and no more.*

Taking the leading coefficient to be +ive, the equation must have a set of +ive terms followed by a set of −ive terms (*i.e.* last term −ive) since there is only one change of sign.

∴ $f(\infty)$+ive and $f(0)=$−ive *i.e.* $f(0)$ and $f(\infty)$ are of opposite signs and as such there must lie at least one or an odd number of roots of the equation $f(x)=0$ between 0 and ∞. But as there is only one change of sign in $f(x)$, the number of +ive roots cannot be greater than one. Hence the equation **must have only** one +ive root.

4. *If all the terms of an equaiion are +ive and the equation involves no odd powers of x, then all its roots are complex.*

Clearly both $f(x)$ and $f(-x)$ will have no change of signs and hence by Descarte's Rule there will not be any +ive or −ive roots. Therefore all the roots must be complex.

5. *If all the terms of an equation are +ive and all involve odd powers of x, then 0 is is the only real root.*

Follows from above.

Ex. 1. *Apply Descarte's Rule of Signs to discuss the nature of the roots of the equation*

$$x^4+15x^2+7x-11=0. \qquad \textbf{(Pb. 50)}$$

$f(x)=x^4+15x^2+7x-11=0$. It has only one change of sign and hence it must have one +ive root.

[Deduction 3]

$f(-x)=x^4+15x^2-7x-11=0$. As above it must have one +ive root *i. e.* $f(x)=0$ must have one negative root. Thus the equation has two real roots, one +ive and one −ive and hence the other two roots must be imaginary.

Ex. 2. *Show that the quintic* $x^5+x^3-8x-5=0$ *cannot have more than three real roots and prove that it must have three real roots.* **(Pb. 33)**

$f(x)=0$ has only one change of sign and hence the equation must have one +ive root. **[Deduction 3]**

$f(-x)=-x^5-x^3+8x-5=0$ has two changes of sign and as such it can have at the most two +ive roots or the max. number of −ive roots of $f(x)=0$ is two.

Again $f(0)=-$ive, $f(-1)=+$ive, $f(-\infty)=-$ive.

Since $f(0)$ and $f(-\infty)$ are of the same sign, so there lies either no or an even number of roots between 0 and $-\infty$ and now $f(0)$ and $f(-1)$ are of opposite signs and also $f(-1)$ and $f(-\infty)$ too are of opposite signs and hence one −ive root lies between 0 and −1 and the other between −1 and $-\infty$ [§ 2]. Thus we conclude that the equation must have three real roots and hence two complex.

Ex. 3. *Show that the equation* $x^{12}-x^4+x^3-x^2+1=0$ *has at least six complex roots.* **(Pb. 31)**

$f(x)=0$ has four changes and hence it can at the most have four +ive roots.

$f(-x)=x^{12}-x^4-x^3-x^2+1=0$ has two changes and it can at the most have two +ive roots or $f(x)=0$ can at the most have two −ive roots. Hence the max. number of real roots of $f(x)=0$ can be six, but there being twelve roots, the min. number of complex roots must be six.

Ex. 4. *Show that the equation* $x^7-3x^4+12x^2+5x-4=0$ *has at least two imaginary roots.*

Ex. 5. *Prove that the equation* $x^5-x+16=0$ *has two pairs of complex roots.* **(Delhi Hon's 51)**

Ex. 6. *Show that the equation* $x^3-qx+r=0$ *where* q *and* r *are essentially positive has one negative root and that the other two* *roots are either imaginary or both +ive.*

$f(x)=0$ has got only two changes of sign and hence the number of +ive roots cannot exceed two.

$$f(-x)=-x^3+qx+r=0 \quad \text{or} \quad x^3-qx-r=0$$

and it has only one change of sign; hence by deduction (3) $f(x)=0$ must have one −ive root. Otherwise also, by deduction I, the equation must have one real root whose sign is opposite to that of the last term and hence it should be −ive.

Hence we conclude that one root of $f(x)=0$ is essentially −ive and therefore the remaining two are either both +ive or imaginary.

Ex. 7. *Find the equation whose roots are the squares of the roots of the equation* $x^3+4x^2+9x+10=0$ *and hence find the nature of the roots of the given equation.*

Ans. One −ive, two complex.

§ 3. Multiple roots.

If an equation $f(x)=0$ has exactly m roots equal to α, then $f(x)$ and its first $(m-1)$ derivatives all vanish for $x=\alpha$; but the mth and all the following derivatives do not vanish.

Conversely—*If $f(x)$ and its first $(m-1)$ derivatives vanish for $x=\alpha$, then $f(x)=0$ has m roots each equal to α.*

Proof—Let $\qquad f(x)=(x-\alpha)^m\phi(x) \qquad \ldots \quad \ldots \quad \ldots(1)$

where $\phi(x)$ does not vanish for $x=\alpha$ for if it vanishes, then it will contain the factor $x-\alpha$ and then $f(x)$ will have more than m equal roots. Differentiating both sides of (1), we get

$$f'(x)=m(x-\alpha)^{m-1}\phi(x)+(x-\alpha)^m\phi'(x)$$

or $$f'(x)=(x-\alpha)^{m-1}[m\phi(x)+(x-\alpha)\phi'(x)]$$

or $$f'(x)=(x-\alpha)^{m-1}\psi_1(x) \qquad \ldots \quad \ldots \quad \ldots(2)$$

where $\psi_1(x)$ does not vanish for $x=\alpha$, because $\psi_1(\alpha)=m\phi(\alpha)$ and it can be zero only if $\phi(\alpha)=0$ which is contrary to the supposition. From (2), we observe that a root which occurs exactly m times in $f(x)=0$ occur exactly $(m-1)$ times in $f'(x)=0$. Hence we have the following :—

A multiple root of order m of the equation $f(x)=0$ is a multiple root of order $(m-1)$ of the first derived equation $f'(x)=0$ and hence $f(x)$ and $f'(x)$ have the common factor $(x-\alpha)^{m-1}$.

Again if we differentiate (2), we get

$$f''(x)=(x-\alpha)^{m-2}.[(m-1)\,\psi_1(x)+(x-\alpha)\,\psi_1'(x)]$$
$$=(x-\alpha)^{m-2}\psi_2(x),$$

where $\psi_2(x)$ does not vanish for $x=\alpha$.

i.e. $f''(x)=0$ has exactly $(m-2)$ equal roots α. Similarly we can show $f'''(x)$ will have the factor $(x-\alpha)^{m-3}$ and so on till $f^{m-1}(x)$ will have the factor $(x-\alpha)$ but $f^m(x)$ will be free from this factor. Thus we have the following :—

Any root which occurs m times in $f(x)=0$ occurs in degrees of multiplicity diminishing by unity in the first derived function.

Converse—$f(x)=f\{\alpha+(x-\alpha)\}$.

Expanding the R. H. S. by Taylor's theorem, we get

$$f(x)=f(\alpha)+(x-\alpha)f'(\alpha)+\frac{(x-\alpha)^2}{2\,!}f''(\alpha)+\dots$$
$$+\frac{(x-\alpha)^{m-1}}{(m-1)\,!}f^{m-1}(\alpha)+\frac{(x-\alpha)^m}{m\,!}f^m(\alpha)$$
$$+\dots\frac{(x-\alpha)^n}{n\,!}f^n(\alpha).$$

Now $f(\alpha)=f'(\alpha)=f''(\alpha)=\dots=f^{m-1}(\alpha)=0$ (given).

$$\therefore\quad f(x)=\frac{(x-\alpha)^m}{m\,!}f^m(\alpha)+\dots+\frac{(x-\alpha)^n}{n\,!}f^n(\alpha).$$

Above shows that $(x-\alpha)^m$ is a factor of $f(x)=0$ which therefore has m roots equal to α.

Determination of Multiple Roots.

We have seen above that multiple roots of order m is a multiple root of order $(m-1)$ of the equation $f'(x)=0$. In order to find such roots we should find the H. C. F. of $f(x)$ and $f'(x)$. This H. C. F. will give the multiple root of $f(x)$ each repeated $(m-1)$ times. Thus if $(x-2)$ is the H. C. F. of $f(x)$ and $f'(x)$, then $f(x)$ contains $(x-2)^2$, as a factor or 2 is a double root of $f(x)=0$. If $(x-2)^2(x-1)^5$ is the H. C. F. of $f(x)$ and $f'(x)$, then $f(x)=0$ has three roots equal to 2 and six roots equal to 1.

Note. In case $f(x)$ and $f'(x)$ have no common factor, then clearly $f(x)=0$ has no equal roots.

Condition for two or three equal roots.

If α be a double root of $f(x)=0$, then $f(\alpha)=0$ and $f'(\alpha)=0$ and the required condition is obtained by eliminating α between $f(\alpha)=0$ and $f'(\alpha)=0$. Similarly if α is a triple root, the required condition can be obtained by eliminating α between $f(\alpha)=0, f'(\alpha)=0, f''(\alpha)=0$.

Ex. 1. *Find the condition that the cubic*

$$a_0x^3+3a_1x^2+3a_2x+a_3=0$$

may have two roots equal and find its value. Also find the condition that the cubic may have all the three roots equal.

The above cubic can be reduced to the form

$$f(z)=z^3+3Hz+G=0 \text{ where } z=a_0x+a_1,$$

$$f_1(z)=3(z^2+H)=0 \quad\text{or}\quad z^2+H=0.$$

The required condition is obtained by eliminating z between $f(z)=0$ and $f_1(z)=0$.

Multiplying $f_1(z)$ by z and subtracting it from $f(z)$, we get $2Hz+G=0$ or $z=-\frac{G}{2H}$. Substituting in $z^2+H=0$, we get $G^2+4H^3=0$ as the required condition. Under the condition the H. C. F. of $f(z)$ and $f_1(z)$ is found to be $2Hz+G$ and hence the equal roots of $f(z)$ are given by $2Hz+G=0$.

$$\therefore \quad a_0x+a_1=z=\frac{-G}{2H}.$$

For three roots equal, eliminate z between

$$f(z)=0,\ f_1(z)=0 \text{ and } f_2(z)=0.$$

But $f_2(z)=0$, gives $z=0$ and hence from $f_1(z)=0$, we get $H=0$ and from $f(z)=0$, we get $G=0$.

$\therefore$ $G=0$, $H=0$ are the required conditions.

Ex. 2. (a) *Show that the equation* $x^n-nax+(n-1)b=0$ *will have a pair of equal roots if* $a^n=b^{n-1}$.

(b) *The equation* $x^n-qx^{n-m}+r=0$ *has two equal roots if*

$$\left\{\frac{q}{n}(n-m)\right\}^n=\left\{\frac{r}{m}(n-m)\right\}^m$$

$$f(x)=x^n-qx^{n-m}+r=0 \quad \ldots \quad \ldots \quad \ldots(1)$$

and

$$f'(x)=nx^{n-1}-q(n-m)x^{n-m-1}=0 \quad \ldots \quad \ldots(2)$$

The required condition is obtained by eliminating x between (1) and (2).

From (2), $n=\dfrac{q(n-m)}{x^m}$ or $x^m=\dfrac{q}{n}(n-m)$ $\ldots \quad \ldots(3)$

and from (1) with the help of (3), we get

$$x^n\left[1-\frac{n}{n-m}\right]+r=0$$

or

$$x^n=\frac{r}{m}(n-m) \quad \ldots \quad \ldots \quad \ldots \quad \ldots(4)$$

or

$$(x^m)^n=(x^n)^m \text{ etc. from (3) and (4).}$$

(c) *If the equation* $x^4-4p^3x+1=0$ *has a pair of equal roots, find the value of* p *and solve the equation completely.*

$f(x)=x^4-4p^3x+1=0$ and $f'(x)=4(x^3-p^3)=0$ giving

$x=p$. Eliminating x between $f(x)=0$ and $f'(x)=0$, we get

$$p^4-4p^4+1=0 \quad \text{or} \quad 3p^4=1 \quad \text{or} \quad p=3^{-1/4}\ldots \qquad \ldots(1)$$

Now let us find the H.C.F. of $f(x)$ and $f'(x)$.

$$\begin{array}{r|l} x^3-p^3) & \overline{x^4-4p^3\ x+1}\ (x \\ & x^4-\ p^3x \\ & \overline{\quad -3p^3x+1.} \end{array}$$

or $$-3p^4\ x+p \quad \text{or} \quad -x+p, \because 3p^4=1 \text{ by (1)}$$

or $$\begin{array}{r|l} x-p) & \overline{x^3-p^3}\ (x^2+px+p^2 \\ & x^3-p \\ & \overline{\quad \times \quad} \end{array}$$

Then the H. C. F. is $(x-p)$ and hence the two equal roots are p, p, their sum being $2p$ and product p^2. Let the other roots be α and β.

Now from the given equation sum of all the roots$=0$.

$$\therefore \quad \alpha+\beta+p+p=0 \quad \text{or} \quad \alpha+\beta=-2p$$

and $$\alpha\,\beta\,p\,p=\text{product of all the roots}=1.$$

$$\therefore \quad \alpha\beta=\frac{1}{p^2}=3p^2 \text{ from (1).}$$

$$\because \quad 3p^4=1.$$

$\therefore$ α and β are the roots of the quadratic

$$t^2+2pt+3p^2=0$$

or $$t=\frac{-2p\pm\sqrt{(4p^2-12p^2)}}{2}=-p+ip\sqrt{2}$$

Hence the four roots are $p,\ p,\ -p\pm ip\sqrt{2}$

where $$x=3^{-(1/4)}.$$

Ex. 3. *Show that the equation*

(i) $x^n-1=0$.

(ii) $$1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\ldots+\frac{x^n}{n!}=0$$

cannot have equal roots.

Hint. (ii) Here $f(x)=e^x$ and $f'(x)=e^x$; they have no H. C. F. which is a function of x and hence no equal roots.

Ex. 4. *If the equation* $x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=0$ *has two roots equal to* α, *prove that* α *is also a root of the equation*

$$a_1x^{n-1}+2a_2x^{n-2}+3a_3x^{n-3}+\ldots\ldots+na_n=0.$$

Since α is a double root, it satisfies both $f(x)=0$ and $f'(x)=0$ and hence will satisfy $nf(x)-xf'(x)=0$ etc.

Ex. 5. *Prove that the equation* $x^5+5ax^3+5a^2x+b=0$ *has a pair of equal roots when* $b^2+4a^5=0$ *and that if it has one pair of equal roots, it must have a second pair.*

The required condition is obtained by eliminating x between $f(x)=0$ and $f'(x)=0$.

$$f(x)=x^5+5ax^3+5a^2x+b=0 \quad \ldots \quad \ldots \quad (1)$$

$$f'(x)=x^4+3ax^2+a^2=0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Eliminate x between the above two.

Multiplying (2) by x and subtracting from (1), we get

$$2ax^3+4a^2x+b=0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Multiplying (2) by $2a$ and (3) by x and subtracting, we get

$$2a^2x^2-bx+2a^3=0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Again multiplying (3) by a and (4) by x and subtracting, we get

$$bx^2+2a^3x+ab=0 \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Now multiplying (4) by b and (5) by $2a^2$ and subtracting, we get $-(b^2+4a^5)x=0$ or $b^2+4a^5=0$ is the required condition. If we find the H. C. F. of $f(x)$ and $f'(x)$, then under the condition $b^2+4a^5=0$ it comes out to be

$$2a^2x^2-bx+2a^3=0$$

and hence $f(x)$ will have a factor $(2a^2x^2-bx+2a^3)^2$ and as such if $f(x)=0$ has a pair of equal roots it must have another pair.

Ex. 6. (a) *Prove that if the coefficients of a given equation are all integers, an integral root is an exact divisor of the absolute term.*

Let the equation be

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0$$

where all the a's are integers. If α be an integral root of above equation then

$$a_0\alpha^n+a_1\alpha^{n-1}+a_2\alpha^{n-2}+\ldots+a_{n-1}\alpha+a_n=0$$

or $$\alpha(a_0\alpha^{n-1}+a_1\alpha^{n-2}+\ldots+a_{n-1})=-a_n$$

or $$\frac{a_n}{\alpha}=-(a_0\alpha^{n-1}+a_1\alpha^{n-2}+\ldots+a_{n-1})=\text{an integer}$$

Hence $\frac{a_n}{\alpha}$=an integer. Therefore the integral root α divides a_n exactly.

(b) *Find the values of a, for which the equation*

$$ax^3-9x^2+12x-5=0$$

has equal roots and solve the equation completely in one case.

$$f(x)=ax^3-9x^2+12x-5=0 \quad \ldots \quad \ldots \quad (1)$$

$$f'(x)=3(ax^2-6x+4)=0 \quad \text{or} \quad ax^2-6x+4=0 \quad \ldots \quad (2)$$

Multiplying (2) by x and subtracting from (1), we get

$$3x^2-8x+5=0 \quad \ldots \quad \ldots \quad \ldots (3)$$

Solving (2) and (3) by the method of cross-multiplication,

$$\frac{x^2}{-30+32}=\frac{x}{12-5a}=\frac{1}{18-8a}$$

Eliminating x, we get

$$(12-5a)^2=2(18-8a) \quad \text{or} \quad 25a^2-104a+108=0$$

or $$(a-2)(25a-54)=0, \quad \therefore \quad a=2 \quad \text{or} \quad \tfrac{54}{25}.$$

Putting $a=2$, we get

$$f(x)=2x^3-9x^2+12x-5=0$$

$$f'(x)=x^2-3x+2=0 \quad \text{or} \quad (x-1)(x-2)=0$$

whose roots are 1 and 2. Either can be a double root of the given equation $f(x)=0$. Since (2) does not divide the absolute term of the given equation, as such it cannot be its root. Hence (1) is a double root of the given equation corresponding to value 2 of a.

Ex. 7. (a) *If $x^5+5px^3+5qx^2+r=0$ has two equal roots, prove that either of them is a root of the quadratic*

$$3qx^2-6p^2x-4pq+r=0.$$

(b) *Prove that the condition for the equation* $x^5-10a^3x^2+b^4x+c^5=0$ *to have three equal roots is*

$$ab^4-9a^5+c^5=0.$$

Eliminate x between $f(x)=0, f_1(x)=0$ and $f_2(x)=0$.

The last of these gives $x=0$ which when substituted in $f(x)=0$ gives the required condition.

Ex. 8. *If the equation* $x^4+ax^3+bx^2+cx+d=0$ *has three equal roots, show that each of them is* $\dfrac{(6c-ab)}{(3a^2-8b)}$.

In the process of finding H. C. F. of $f(x)$ and $f'(x)$, the 1st quotient is $x+a$ and the quadratic remainder is $F(x)=(8b-3a^2)\,x^2+2\,(6c-ab)\,x+(16d-ac)$. If the equation is to have three equal roots, then the H. C. F. should be perfect square of a linear expression in x. Hence $F(x)$ should be the H. C. F. and must have equal roots, the condition for which is

$$4\,(6c-ab)^2=4(8b-3a^2)\,(16d-ac).$$

Multiplying $F(x)$ by $8b-3a^2$, it becomes, with the help of the above condition,

$$[(8b-3a^2)\,x+(6c-ab)]^2$$

and hence the equal root is given by

$$F(x)=0 \quad \text{or} \quad x=\frac{6c-ab}{3a^2-8b}.$$

Ex. 9. *Find the multiple roots of the following equations :—*

(i) $x^4+3x^3-7x^2-15x+18=0$. **Ans.** -3.

(ii) $x^5-x^4+2x^3-2x^2+x-1=0$. **Ans.** $\pm i$.

Find the H. C. F. of $f(x)$ and $f'(x)$.

Ex. 10. *Solve the equations*

(i) $x^4-8x^3+24x^2-32x+16=0$,

(ii) $x^6-6x^4-4x^3+9x^2+12x+4=0$,

being given that they have equal roots.

(i) All the roots equal to 2.

(ii) 2 roots equal to 2 and 4 roots equal to -1.

Ex. 11. *Find the conditions that a biquadratic*

$$f(x)\equiv a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$$

may have three roots equal and show that in this case it can be expressed in the form

$$a_0^3f(x)=[a_0x+a_1+\sqrt{(-H)}]^3\,[a_0x+a_1-3\sqrt{(-H)}].$$

The given equation can be reduced to the form

$F(z)\equiv z^4+6Hz^2+4Gz+a_0^2I-3H^2=0$ where $z=a_0x+a_1$,

$F_1(z)=z^3+3Hz+G=0$.

Now if $F(z)=0$ has three equal roots then $F_1(z)=0$ will have two equal roots, the condition for which is

$$G^2+4H^3=0 \quad \text{[by Ex. I P. 159]} \ldots \ \ldots \ \ldots (1)$$

Now finding the H. C. F. of $F(z)$ and $F_1(z)$, we get the first quotient as z and the quadratic remainder is $\phi(z)=3Hz^2+3Gz+a_0^2I-3H^2=0$. If $F(z)$ is to have three equal roots then the quadratic should be the H. C. F. and it should be a perfect square, the condition for which is

$$9G^2-12H\,(a_0^2I-3H^2)=0$$

or $\quad 9\,(G^2+4H^3)-12Ha_0^2I=0$ or $-12Ha_0^2I=0$ from (1).

$\therefore$ $I=0$ because H cannot be zero as in that case $\phi(z)$ is reduced to linear.

Also $a_0^2\,(HI-a_0^3J)=G^2+4H^3=0$; $\quad\therefore\ J=0$.

$\therefore$ $I=0$ and $J=0$ are the required conditions and the condition $G^2+4H^3=0$ is included in them. With the help of this condition, we get the H. C. F.

$$\phi\,(z)=3Hz^2+3Gz-3H^2=0 \qquad [\because\ I=0]$$

or $\quad H^2z^2+GHz-H^3=0.$ Put $G^2=-4H^3$

or $\quad H^2z^2+GHz+\dfrac{G^2}{4}=0$ or $\left(Hz+\dfrac{G}{2}\right)^2=0.$

Hence each of the triple roots is

$$z=\frac{-G}{2H}=-\frac{\sqrt{(-4H^3)}}{\sqrt{(4H^2)}}=-\sqrt{(-H)}$$

If the fourth root be α, then sum of the roots is

$$\alpha-3\sqrt{(-H)}=0, \quad \therefore \quad \alpha=3\sqrt{(-H)}.$$

Hence $\quad F(z)\equiv[z+\sqrt{(-H)}]^3\,[z-3\sqrt{(-H)}]$

or $\quad a_0^3 f(x)=[a_0x+a_1+\sqrt{(-H)}]^3\,[a_0x+a_1-3\sqrt{(-H)}].$

Ex. 12. *Find the conditions which must be satisfied when the biquadratic of the last example be a perfect square and prove that in that case* $a_0^3 f(x)=\{(a_0x+a_1)^2+3H\}^2$. **(Pb. 60, Agra 39)**

In this case H.C.F. $\phi(z)$ of last example should be a quadratic (but not perfect square).

$\therefore \quad 3Hz^2+3Gz+(a_0^2I-3H^2)=0$ should be the H.C.F.

or $$z^2+\frac{G}{H}z+\frac{a_0^2I-3H^2}{3H}=0$$

and hence

$$z^4+6Hz^2+4Gz+a_0^2I-3H^2\equiv\left[z^2+\frac{G}{H}z+\frac{a_0^2I-3H^2}{3H}\right]^2.$$

Comparing the coefficients of like powers of z, we get

$$\frac{2G}{H}=0, \quad \therefore \quad G=0$$

and $$\frac{G^2}{H^2}+\frac{2(a_0^2I-3H^2)}{3H}=6H. \quad \text{Put} \quad G=0.$$

$$\therefore \quad a_0^2I-3H^2=9H^2 \quad \text{or} \quad a_0^2I=12H^2.$$

Under these conditions the H. C. F. is reduced to (z^2+3H) and hence

$$F(z)=(z^2+3H)^2. \quad \text{Put} \quad z=a_0x+a_1$$

or $$a_0^3 f(x)=[(a_0x+a_1)^2+3H]^2.$$

Note—See also Q. 3 P. 134.

Ex. 13. *Show that* $a\alpha+b=\dfrac{-GI}{2HI-3aJ}$ *or* $-\sqrt{(-H)}$ *according as* α *is a double or triple root of the biquadratic of the last example.* **(Agra 42)**

The 2nd part is done in Ex. 11. For the first part find the H. C. F. of $F(z)$ and $F'(z)$ and if the biquadratic has two equal roots, the H. C. F. should be linear and it comes out to be $(2HI-3aJ)z+GI=0$ giving the value of z and hence of x etc. Relation $G^2+4H^3=a_0^2(HI-a_0J)$ will be used.

Ex. 14. *If* $\alpha, \beta, \gamma, \delta\ldots$ *be the roots of* $f(x)=0$ *and* $\alpha', \beta', \gamma'\ldots$ *of* $f'(x)=0$, *prove that*

$$f'(\alpha)f'(\beta)f'(\gamma)f'(\delta)\ldots=n^n f(\alpha')f(\beta')f(\gamma')\ldots.$$

and that each is equal to the absolute term in the equation whose roots are the squares of the difference of the roots of $f(x)=0$.

(Agra 51)

Let $f(x)=0$ be an equation of nth degree so that

$$f(x)=(x-\alpha)(x-\beta)(x-\gamma)\ldots n \text{ factors} \quad \ldots \quad \ldots(1)$$

or say $$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots.$$

then $$f'(x)=nx^{n-1}+(n-1)p_1x^{n-2}+\ldots$$

But we are given that the roots of

$$f'(x)=0 \text{ are } \alpha', \beta', \gamma'\ldots$$

$$\therefore\quad f'(x)=K(x-\alpha')(x-\beta')(x-\gamma')\ldots(n-1) \text{ factors.}$$

Comparing the coefficients of x^{n-1} in the two expressions for $f'(x)$, we find that $K=n$.

$$\therefore\quad f'(x)=n(x-\alpha')(x-\beta')(x-\gamma')\ldots(n-1) \text{ factors} \quad \ldots(2)$$

Putting in (2) the n quantities α, β, γ, successively for x, we get

$$f'(\alpha)=n(\alpha-\alpha')(\alpha-\beta')(\alpha-\gamma')\ldots(n-1) \text{ factors,}$$

$$f'(\beta)=n(\beta-\alpha')(\beta-\beta')(\beta-\gamma')\ldots(n-1) \text{ factors}$$

....

....

n such equations as $\alpha, \beta, \gamma\ldots$ are n in number. Multiplying the above vertically,

$$f'(\alpha)f'(\beta)f'(\gamma)\ldots\ldots\ldots n \text{ factors}$$

$$=n^n\left\{\begin{array}{l}(\alpha-\alpha')(\beta-\alpha')(\gamma-\alpha')\ldots\ldots n \text{ factors}\\ (\alpha-\beta')(\beta-\beta')(\gamma-\beta')\ldots\ldots n \text{ factors}\\ (\alpha-\gamma')(\beta-\gamma')(\gamma-\gamma')\ldots\ldots n \text{ factors}\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ (n-1) \text{ such factors as } \alpha', \beta', \gamma' \text{ are}\\ (n-1) \text{ in number.}\end{array}\right\}\ldots\ \ldots(3)$$

Again putting the $(n-1)$ quantities α', β', γ'.....respectively for x in (1), we get

$$\left.\begin{array}{l}f(\alpha')=(-1)^n(\alpha-\alpha')(\beta-\alpha')(\gamma-\alpha')\ldots\ldots n \text{ factors}\\ f(\beta')=(-1)^n(\alpha-\beta')(\beta-\beta')(\gamma-\beta')\ldots\ldots n \text{ factors}\\ f(\gamma')=(-1)^n(\alpha-\gamma')(\beta-\gamma')(\gamma-\gamma')\ldots\ldots n \text{ factors}\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ (n-1) \text{ such equations.}\end{array}\right\}\ldots(4)$$

Hence from (3) and (4), we get

$$\therefore\quad f(\alpha')f(\beta')f(\gamma')\ldots\ldots(n-1) \text{ factors}$$

$$=\frac{(-1)^{n(n-1)}}{n^n}\{f'(\alpha)f'(\beta)f'(\gamma)\ldots.n \text{ factors}\}.$$

Now $n(n-1)$ being the product of two consecutive numbers must be even and as such $(-1)^{n(n-1)}=1$.

$$\therefore\quad n^nf(\alpha')f(\beta')f(\gamma')\ldots\ldots$$

$$=f'(\alpha)f'(\beta)f'(\gamma)\ldots\ldots$$

Now we have to prove that

$f'(\alpha)f'(\beta)f'(\gamma)\ldots\ldots.=$absolute term in the equation whose roots are the squares of the difference of the roots of the equation $f(x)=0$.

$$f(x)=(x-\alpha)(x-\beta)(x-\gamma)\ldots\ldots$$

$$\log f(x)=\log(x-\alpha)+\log(x-\beta)+\log(x-\gamma)\ldots\ldots$$

Differentiating

$$f'(x)=\frac{f(x)}{(x-\alpha)}+\frac{f(x)}{(x-\beta)}+\frac{f(x)}{(x-\gamma)}+\cdots$$

$$=(x-\beta)(x-\gamma)\ldots+(x-\alpha)(x-\gamma)\ldots$$
$$+(x-\alpha)(x-\beta)\ldots+\ldots$$

Putting $x=\alpha, \beta, \gamma\ldots$, we get

$$f'(\alpha)=(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\ldots$$
$$f'(\beta)=(\beta-\alpha)(\beta-\gamma)(\beta-\delta)\ldots$$
$$f'(\gamma)=(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)\ldots$$

$\therefore\ f'(\alpha)f'(\beta)f'(\gamma)\ldots$

$$=(-1)^{n(n-1)/2}(\alpha-\beta)^2(\beta-\gamma)^2(\alpha-\gamma)^2\ldots$$

= product of all the roots of the equation of squared difference

= absolute term (numerically) of the equation of squared difference. Hence proved

Ex. 15. *If $\alpha, \beta, \gamma, \delta$ be the roots of $f(x)=0$, prove that $f'(\alpha)+f'(\beta)+f'(\gamma)+f'(\delta)$ can be expressed as the product of three factors.*

Just as in last example, 2nd part,

$$f'(\alpha)+f'(\beta)+f'(\gamma)+f'(\delta)$$
$$=(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)+(\beta-\alpha)(\beta-\gamma)(\beta-\delta)$$
$$+(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)+(\delta-\alpha)(\delta-\beta)(\delta-\gamma)$$
$$=(\alpha-\beta)\{(\alpha^2-\gamma\alpha-\delta\alpha+\gamma\delta)-(\beta^2-\gamma\beta-\delta\beta+\gamma\delta)\}$$
$$+(\gamma-\delta)\{(\gamma^2-\alpha\gamma-\beta\gamma+\alpha\beta)-(\delta^2-\alpha\delta-\beta\delta+\alpha\beta)\}$$
$$=(\alpha-\beta)\{\alpha^2-\beta^2-(\alpha-\beta)(\gamma+\delta)\}$$
$$+(\gamma-\delta)\{(\gamma^2-\delta^2)-(\gamma-\delta)(\alpha+\beta)\}$$
$$=(\alpha-\beta)^2(\alpha+\beta-\gamma-\delta)-(\gamma-\delta)^2(\alpha+\beta-\gamma-\delta)$$
$$=(\alpha+\beta-\gamma-\delta)\{(\alpha-\beta)^2-(\gamma-\delta)^2\}$$
$$=(\alpha+\beta-\gamma-\delta)(\alpha-\beta-\gamma+\delta)(\alpha-\beta+\gamma-\delta)$$
$$=(\alpha+\beta-\gamma-\delta)(\alpha+\gamma-\beta-\delta)(\alpha+\delta-\beta-\gamma)$$

§4. Maximum and Minimum Values of f (x).

We know from the definition of Max. and Min. that as x varies in the interval $(x-h,\ x+h)$, where h is small, then $f(x)$ is greater than both $f(x-h)$ and $f(x+h)$ if $f(x)$ be max. *i.e.* $f(x-h)-f(x)$ and $f(x+h)-f(x)$ are of the same sign *i.e.* $-$ive. Similarly if $f(x)$ is Min., then $f(x)$ is less than both $f(x-h)$ and $f(x+h)$ *i.e.* $f(x-h)-f(x)$ and $f(x+h)-f(x)$ are of the same sign *i.e.* $+$ive.

Now by Taylor's Theorem,

$$\left.\begin{aligned} &f(x+h)-f(x)\\ &\quad=hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f''''(x)+\dots\dots\\ &f(x-h)-f(x)\\ &\quad=-hf'(x)+\frac{h^2}{2!}f''(x)-\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f''''(x)-\dots\dots \end{aligned}\right\}\dots(1)$$

Now when h is made sufficiently small the sign of the right hand side of each equation in (1) and therefore of the left hand side is ultimately dependent upon that of $hf'(x)$, that being the term of lowest degree in h. But both $f(x+h)-f(x)$ and $f(x-h)-f(x)$ are of opposite sign as seen from (1) whereas they should have the same signs if $f(x)$ is either Max. or Min. It is therefore necessary that $f(x)$ should vanish so that the lowest term of the right hand side of equation in (1) should depend upon an even power of h.

Hence $f'(x)=0$ is the essential condition for the occurrence of Max. and Min. values of $f(x)$.

Let the roots of $f'(x)=0$ be a, b, c etc.

Consider one of the roots of the above equation say b; then putting $x=b$ and $f'(x)=0$ at $x=b$ in (1), we get

$$f(b+h)-f(b)=\frac{h^2}{2!}f''(b)$$
$$+\frac{h^3}{3!}f'''(b)+\frac{h^4}{4!}f''''(b)+\ldots,$$

$$f(b-h)-f(b)=\frac{h^2}{2!}f''(b)$$
$$-\frac{h^3}{3!}f'''(b)+\frac{h^4}{4!}f''''(b)+\ldots$$

Now if $f''(b)$ is $-$ive, then clearly $f(b+h)$ and $f(b-h)$ are both less than $f(b)$ so that $f(x)$ is Max. at $x=b$. In case $f''(b)$ is $+$ive, then both $f(b+h)$ and $f(b-h)$ are greater than $f(b)$, showing that $f(x)$ is Min. at $x=b$.

Geometrical Significance.

We know that if $y=f(x)$ be the equation of a curve, then $f'(x)$ *i.e.* $\frac{dy}{dx}$ represents the slope of the tangent at any point on the curve, we have seen above if $f(x)$ be Max. or Min. at any point, then $f'(x)=0$ for that point which means that at every point, where $f(x)$ is Max. or Min., the tangent to the curve $y=f(x)$ is parallel to x-axis.

Note—In case $f''(b)=0$, then $f'''(b)$ **must also be zero** if the function is Max. or Min.; and if $f''''(b)$ is $-$ive, $f(x)$ will be Max.; and if $f''''(b)$ is $+$ive, then $f(x)$ will be Min. at $x=b$.

Working Rule.

Differentiate the given function $f(x)$ and solve $f'(x)=0$. The function will be Max. for those values of x which make $f''(x)$ $+$ive.

(b) Show that the Max. and Min. values of the biquadratic $ax^4+4bx^3+6cx^2+4dx+e$ *are the roots of the equation*

$a^3k^3-3\,(a^2I-9H^2)\,k^2+3\,(aI^2-18HJ)\,k-\Delta=0$ *where* Δ *is the discriminant of the quartic.* **(Agra 38, 41)**

We know from P. 132, that the discriminant Δ of the quartic is I^3-27J^2 where $I=ae-4bd+3c^2$

and $J=ace+2bcd-ad^2-b^2e-c^3.$

Let $f(x)=ax^4+4bx^3+6cx^2+4dx+e.$

Now if the curve $y=f(x)$ be moved parallel to the axis of x through a distance k, where k is the Max. or Min. value of $f(x)$, then x-axis will become a tangent *i.e.* the two values of x will coincide. Now by moving the curve parallel to x-axis through a distance k, y becomes $y-k$. Then the equation $f(x)-k=0$ will have two roots equal, the condition for which is that its discriminant should be zero.

$$f(x)-k\equiv ax^4+4bx^3+6cx^2+4dx+(e-k)=0.$$

Its discriminant is $I'^3-27J'^2$ where in the usual values of I and J we have to put e equal to $e-k$ in order to get I' and J'.

$$\therefore\quad I'=a\,(e-k)-4bd+3c^2=I-ak,$$

$$J'=ac\,(e-k)+2bcd-ad^2-b^2\,(e-k)-c^3$$

$$=J-(ack-b^2k)=J-kH.$$

$$\because\ H=ac-b^2.$$

$\therefore\quad I'^3-27J'^2=0$ gives

$$(I-ak)^3-27\,(J-kH)^2=0$$

or $$I^3-3I^2ak+3Ia^2k^2-a^3k^3-27J^2+54JkH-27k^2H^2=0.$$

Cancelling minus sign and putting $I^3-27J^2=\Delta$, we get

$$a^3k^3-3\,(a^2I-9H^2)\,k^2+3\,(aI^2-18HJ)\,k-\Delta=0.$$

The above equation gives the value of k.

Ex. 2. *Show that the max. and min. values of the cubic* $ax^3+3bx^2+3cx+d$ *are the roots of the equation* $a^2\rho^2-2G\rho+\Delta=0$ *where* Δ *is the discriminant.*

Discriminant of a cubic is $G^2+4H^3=\Delta$. [§ 13 P. 79]

§5. Rolle's Theorem. *Between two consecutive real roots of the equation $f(x)=0$ there lies at least one real root of the equation $f'(x)=0$.*

Now a and b are consecutive roots of $f(x)=0$ and as x varies from a to b, $f(x)$ varies continuously from $f(a)$ to $f(b)$. It will therefore vary either by increasing first and then decreasing or by decreasing first and then increasing. It must therefore pass through at least one (in general odd) Max. or Min. values during its variation from $f(a)$ to $f(b)$ and let α be the value of x between a and b where $f(x)$ is either Max. or Min. and from the definition of Max. or Min. $f'(x)=0$ at $(x=\alpha)$. Thus a number α lying between a and b is a root of the equation $f'(x)=0$.

Analytical Proof of Rolle's Theorem.

Let us suppose that the consecutive roots a and b are multiple roots of order m and n respectively and the product of the factors corresponding to the other roots of $f(x)=0$ be denoted by $\phi(x)$.

$$\therefore \quad f(x)=(x-a)^m (x-b)^n \phi(x) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots(1)$$

$$f'(x)=m(x-a)^{m-1}(x-b)^n\phi(x)+n(x-a)^m(x-b)^{n-1}\phi(x)$$
$$+(x-a)^m(x-b)^n\phi'(x)$$

or
$$f'(x)=(x-a)^{m-1}(x-b)^{n-1}[m(x-b)\phi(x)+n(x-a)\phi(x)$$
$$+(x-a)(x-b)\phi'(x)]$$

or
$$f'(x)=(x-a)^{m-1}(x-b)^{n-1}\psi(x) \ldots \quad \ldots \quad \ldots(2)$$

where
$$\psi(x)=m(x-b)\phi(x)+n(x-a)\phi(x)+(x-a)(x-b)\phi'(x) \ldots(3)$$

$$\psi(a)=m(a-b)\phi(a),$$

$$\psi(b)=n(b-a)\phi(b).$$

But $\phi(a)$ and $\phi(b)$ are of the same sign for if they are of opposite signs then a root of $\phi(x)=0$ and hence of $f(x)=0$ will lie between a and b [§ 2] which will contradict the supposition that a and b are consecutive roots of

$f(x)=0$. Hence $\psi(a)$ and $\psi(b)$ are clearly of opposite signs and as such there lies at least one (in general odd) number of roots of the equation $\psi(x)=0$ and hence of $f'(x)=0$ [from (3)] between a and b. Thus the theorem is proved.

Deduction—*To prove that between the two consecutive roots of the equation $f'(x)=0$ there lies at the most one real root of the equation $f(x)=0$ or there may not lie any.*

Let x_1 and x_2 be the two consecutive roots of the equation $f'(x)=0$ lying between the intervals (α, β) and (β, γ) where α, β, γ are three consecutive roots of the equation $f(x)=0$. **[Rolle's Theorem]**

Hence $\alpha < x_1 < \beta < x_2 < \gamma$. This relation shows that β, a root of $f(x)=0$, lies between x_1 and x_2, the consecutive roots of $f'(x)=0$.

To prove that there cannot lie more than one root.

If possible, let us suppose that there are two roots β and β' of the equation $f(x)=0$ which lie between x_1 and x_2, the consecutive roots of $f'(x)=0$, so that $x_1 < \beta < \beta' < x_2$; then β and β' being consecutive roots of $f(x)=0$ must have between them a root of the equation $f'(x)=0$ [Rolle's Theorem] which is contrary to the supposition that x_1 and x_2 are consecutive roots of $f(x)=0$. Hence there cannot lie more than one root of $f(x)=0$ between two consecutive roots of the equation $f'(x)=0$. In this case $f(x_1)$ and $f(x_2)$ should be of opposite signs and if they are of the same sign then no root of $f(x)=0$ will lie between x_1 and x_2.

Ex. 1. *(a) The equation $x^4-15x^3+75x^2-145x+84=0$ has its roots 1, 3, 4, 7. Locate the roots of the equation*

$$4x^3-45x^2+150x-145=0.$$

If $f(x)=x^4-15x^3+75x^2-145x+84=0$,

then $$f'(x)=4x^3-45x^2+150x-145=0$$

Now the roots of $f(x)=0$ are 1, 3, 4, 7 arranged in order and by Rolle's Theorem we know that between two consecutive roots of the equation $f(x)=0$ there lies at least one real root of the equation $f'(x)=0$. Hence there will be a root each of $f'(x)=0$ between 1, 3 ; 3, 4 ; and 4, 7.

§ 6. Limits of the roots of an equation.

Definitions.

(a) Superior or upper limit of positive roots :— A number which is greater than all the positive roots of a given equation is called an upper or superior limit of the positive roots of that equation.

(b) Inferior or lower limit of positive roots :—A number which is less than all the +ive roots of a given equation is called the inferior or lower limit of the positive roots of that equation.

(c) Superior or upper limit of negative roots :—A number which is greater than all the negative roots of a given equation is called an upper or supperior limit of the negative roots of that equation. In other words, superior limit of negative roots of a given equation is that negative number below which (numerically) lie all the –ive roots.

(d) Inferior or lower limit of negative roots :—A number which is less than all the negative roots of a given equation is called the inferior or lower limit of negative roots of that equation., In other words, inferior limit of the –ive roots of a given equation is that negative number above which (numerically) lie all the –ive roots.

Combining the above definitions we can say that the superior limit of the real roots of a given equation is that number which is greater than all the real roots of that equation and the inferior limit of the real roots is that number which is less than all the real roots of that equation. Thus superior limit of +ive roots is the superior limit of the real roots and the inferior limit of the −ive roots is the inferior limit of the real roots of a given equation.

Lower limits deduced from upper limits.

(i) If k is the upper limit of the +ive roots of the equation $f(-x)=0$, then evidently $-k$ is the lower limit to the real roots of $f(x)=0$, because −ive roots of $f(x)=0$ are +ive roots of $f(-x)=0$.

(ii) Similarly if k is the upper limit of +ive roots of $f\left(\frac{1}{x}\right)=0$, then $\frac{1}{k}$ is the lower limit of the +ive roots of $f(x)=0$.

(iii) Limits of −ive roots deduced from the limits of +ive roots.

We know that −ive roots of $f(x)=0$ are +ive roots of $f(-x)=0$. Then if h and k are the upper and lower limits of the +ive roots of $f(-x)=0$, then $-h$, $-k$ are the lower and upper limits respectively of the −ive roots of $f(x)=0$.

§ 7. Methods of finding the upper limit of +ive roots.

(a) *The upper limit of +ive roots of $f(x)=0$ in which the coefficient of leading term is unity is equal to the numerically greatest coefficient increased by one.*

(b) Divide the numerical value of each negative coefficient by the sum of all the +ive coefficients which precede that particular −ive coefficient; then the greatest of the fractions thus obtained and increased by unity gives the upper limit of the +ive roots of $f(x)=0$.

This method will give us a closer limit than the former.

(c) Newton's Method—If h is a number which makes $f(x)$ and all its derivatives +ive, then h is an upper limit of the +ive roots of $f(x)=0$ which is also the upper limit of real roots of $f(x)=0$.

(d) Express the equation $f(x)=0$ in the form

$$f(x) \equiv \phi_1(x) - \phi_2(x) + \phi_3(x) = 0,$$

where $\phi_1(x)$ is the sum of all the terms which precede the first −ive term, $-\phi_2(x)$ is the sum of all the −ive terms, and $\psi_3(x)$ is the sum of all other +ive terms. Now if h be the least +ive number which makes $\phi_1(x)-\phi_2(x)$ +ive, then h is called the upper limit of +ive roots of $f(x)=0$.

(e) If the first negative coefficient occurs in $(r+1)$th term and K is the numerical value of the greatest negative coefficient, then $(K)^{1/r}+1$ is the superior limit of +ive roots.

CHAPTER VII

STURM'S THEOREM

§ 1. Sturm's method of finding the exact number of real roots of an equation.

We have seen that by Descarte's rule of signs we cannot get the exact number of real roots of a given equation $f(x)=0$. There are other theorems given by Budan, Newton and Fourier, but they too do not exactly give the number of real roots of a given equation. But Sturm's Theorem as will be discussed below gives us the exact number of real roots of a given equation.

§ 2. Sturm's functions. (Agra 33, 49)

Let $f(x)$ be any function of x of degree n and $f'(x)$ be its first derivative. Divide $f(x)$ by $f'(x)$ and let the remainder **with sign changed** be denoted by $f_2(x)$.

Again divide $f'(x)$ by $f_2(x)$ and let the remainder **with sign changed** be denoted by $f_3(x)$.

Continue the process till you get the last remainder **whose sign is also to be changed.**

The above process is the same as that of finding the H. C. F. of $f(x)$ and $f'(x)$ **with the modification that the sign of each remainder is to be changed before it becomes the divisor.** Also we know that in the process of finding the H. C. F. we can multiply and divide any remainder by any constant before using it as a divisor, but here in the above process we can only multiply or divide the remainder only by a **positive constant** before using it as a divisor or by a polynomial of x which is always +ive for real values of x say of the type x^2+1 or x^4+x^2+1 etc.

The series of functions

$$f(x), f'(x), f_2(x), f_3(x) \ldots\ldots f_r(x)$$

consisting of the given function, its derivative and **remainders with their sign changed** in the process of finding the H. C. F. of $f(x)$ and $f'(x)$ are called **Sturm's functions.** **(Agra 57)**

The functions $f_2(x)$, $f_3(x) \ldots\ldots\ldots\ldots f_r(x)$ are called **auxiliary functions.**

$f(x)=0$ **having equal roots.**

In case $f(x)=0$ has equal roots, then we know that $f(x)$ and $f'(x)$ will have a H. C. F. and hence the last of the Sturm's remainder will be a function of x.

$f(x)=0$ **having no equal roots.**

In this case evidently the last of the Sturmian's remainder will be numerical for if it is some function of x, then it would mean that $f(x)$ has got equal roots. In this case there will be $(n+1)$ Sturm's functions

i.e. $$f(x), f'(x), f_2(x), f_3(x) \ldots\ldots f_n(x).$$

§ **3. Sturm's Theorem. All roots unequal.** *If $f(x)$ is a polynomial and a and b be any two real numbers, then the number of distinct real raots of the equation $f(x)=0$ lying between a and b is exactly equal to the difference between the number of changes of signs when x is put equal to a and the number when x is put equal to b in the $(n+1)$ Sturm's functions $f(x)$, $f'(x)$ $f_2(x)$, $f_3(x) \ldots\ldots\ldots\ldots f_n(x)$ consisting of the given function, its derivative and the $(n-1)$ remainders with their sign changed in the process of finding the H. C. F.*

of $f(x)$ and $f'(x)$.

(Agra 34, 44, 47, 50, 52 54 ; Delhi Hon's. 48 51, 54)

From the definition of Sturm's functions we can establish the following relations between them :—

$$\left.\begin{array}{l} f(x)=q_1f'(x)-f_2(x) \\ f'(x)=q_2f_2(x)-f_3(x) \\ f_2(x)=q_3f_3(x)-f_4(x) \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ f_{r-1}(x)=q_rf_r(x)-f_{r+1}(x) \\ \dots\dots\dots\dots\dots\dots\dots \\ f_{n-2}(x)=q_{n-1}f_{n-1}(x)-f_n(x) \end{array}\right\} \dots \quad \dots \quad \dots(A)$$

Proof: From relations (A) we make the following observations :.

1. As $f(x)=0$ has no equal roots, hence $f(x)$ and $f'(x)$ have no common factors and consequently $f_n(x)$ the last Sturmian function is numerical having a definite sign $+$ or $-$.

2. No two consecutive auxiliary functions vanish for the same value of x. If possible, suppose that when $x=\alpha$, both $f_2(x)$ and $f_3(x)$ vanish, which shows that $f'(x)$ contains the factor $x-\alpha$ and from the first of the relations (A) we find that $(x-\alpha)$ is also a factor of $f(x)$. Thus $(x-\alpha)$ is the H. C. F. of $f(x)$ and $f'(x)$ showing the existence of equal roots which is contrary to the hypothesis.

3. If any of the auxiliary functions vanishes, then the two adjacent functions *i. e.* one which precedes it and the one which follows it must have opposite signs. For suppose that $f_3(x)=0$ when $x=\alpha$; then from relation (A), $f_2(x)=-f_4(x)$ for $x=\alpha$ showing that $f_2(x)$ and $f_4(x)$ have opposite signs for $x=\alpha$.

4. The same reasoning applies if any of the q's vanishes. Since all the functions are polynomials in x, no one can change sign as x increases continuously from a to b, when x assumes a value which causes that function to vanish.

5. Now we shall show that when x in passing from a to b takes a value α such that $f_r(\alpha)=0$, then no change of sign is gained or lost. Now since $f_r(\alpha)=0$, we have from (3), $f_{r-1}(\alpha)$ and $f_{r+1}(\alpha)$ must be of opposite signs. Now as $f_r(x)$ passes through zero, it changes its sign either from $+$ to $-$ or from $-$ to $+$. But $f_{r-1}(x)$, $f_{r+1}(x)$ are continuous at $x=\alpha$ so that each of them has an invariable sign near $x=\alpha$. Thus the three functions $f_{r-1}(x)$, $f_r(x)$ and $f_{r+1}(x)$ will have only one change of sign just before $x=\alpha$ and just after $x=\alpha$ *i. e.* their signs can be either $++-$, $--+$, $+--$, $-++$.

Thus we find that whatever sign we may place between two unlike signs, we have only one change of sign. Hence no change of sign is either lost or gained among Sturm's functions.

In case the value of x *i. e.* α be such that it causes more than one of the functions to vanish, then they cannot be consecutive, for in that case, by (2), the equation $f(x)=0$ will have equal roots.

6. Now we shall show that when x in passing from a to b takes a value α which causes $f(x)$ to vanish *i. e.* α be a root of $f(x)=0$, then a change of sign is lost. Now by Taylor's Theorem,

$$f(\alpha-h)=0-hf'(\alpha)+\frac{h^2}{2!}f''(\alpha)\ldots \quad \because \; f(\alpha)=0.$$

$$f(\alpha+h)=0+hf'(\alpha)+\frac{h^2}{2!}f''(\alpha)\ldots, \because f(\alpha)=0.$$

Now let h be sufficiently small so that the sign of the L.H.S. is made to depend on the first term of R.H.S.

Therefore if $f'(\alpha)$ **be +ive,** then $f(\alpha-h)$ is −ive and $f(\alpha+h)$ is +ive *i.e.* in this case the signs of $f(x)$ and $f'(x)$ will be − + just before $x=\alpha$ and + + just after $x=\alpha$. Thus one change of sign is lost. Again if $f'(\alpha)$ **be −ive,** then $f(\alpha-h)$ is +ive and $f(\alpha+h)$ is −ive *i.e.* in this case the signs of $f(x)$ and $f'(x)$ will be +— just before $x=\alpha$ and − − just after $x=\alpha$ Here also one change of sign is lost. Thus we conclude that when x passes through a root α of the equation $f(x)=0$ one change of sign is lost whether $f'(\alpha)$ be +ive or −ive. Hence the number of variations lost as x goes from a real value a to a real value b is exactly equal to the number of real roots of the equation $f(x)=0$ between **a** and **b.**

II Case. Equal roots—*If $f(x)=0$ be an equation having equal roots and the Sturm's functions be found as $f, f', f_1 \ldots f_r$ the last of these being the H.C.F. of $f(x)$ and $f'(x)$, then the difference between the number of changes of signs when a and b are substituted in the Sturm's function is equal to the number of real roots of the equation $f(x)=0$ which lie between a and b,* **each multiple root being counted once only.**

Let $f(x)=(x-\alpha)^p(x-\beta)^q(x-\gamma)(x-\delta)\ldots\ldots$; then clearly $f(x)$ and $f'(x)$ will have an H.C.F. $(x-\alpha)^{p-1}(x-\beta)^{q-1}$ which may be denoted by H.

This H.C.F. will be a factor of all the Sturm's functions $f, f', f_1, f_2 \ldots f_r$. Again let $\psi(x)$ stand for

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\ldots,$$

then clearly $f(x)=H.\psi(x), f'(x)=H.\psi'(x), f_2(x)=H\psi_2(x)\ldots$ For any value of x, the number of changes of sign in the sequence of f's is the same as that in the sequence of ψ's.

Now $\phi(x)=0$ has all its roots unequal and all its roots are the same as those of $f(x)=0$ only with the change that the multiple roots of $f(x)=0$ occur in $\psi(x)=0$ only once.

Now applying the reasoning of 1st case on $\psi(x)=0$ we can say that difference between the number of changes of signs when x is put equal to a and b in the sequence of ψ's (hence of f's) represents exactly the number of real roots of the equation $\psi(x)=0$ lying between a and b or the number of real roots of $f(x)=0$ that lie between a and b but each multiple root being counted only once.

Certain labour saving devices.

1. When $f(x)=0$ has no repeated root *i.e.* the last Sturmian function be numerical, and we are concerned only with its sign, in order to get its sign we put $f_{n-1}(x)=0$ and find the value of x; then we know that for this value of x, $f_{n-2}(x)$ and $f_n(x)$ must have the opposite signs. Thus if the value of x obtained from $f_{n-1}(x)=0$ makes $f_{n-2}(x)$ +ive, then $f_n(x)$ is −ive and if it makes $f_{n-2}(x)$ −ive, then $f_n(x)$ is +ive. This device saves us the labour of actually calculating the value of $f_n(x)$ (which is numerical).

2. If any of the Sturmian functions say $f_2(x)$ has all its roots imaginary, we may stop further calculation and we should use $f, f', f_1, f_2\ldots\ldots f_r$, functions only for Sturm's Theorem, because in this theorem the last of the

functions should be of invariable sign for all real values of x and we know from the properties of equations that if $f(x)=0$ has all its roots imaginary, then $f(x)$ is always +ive for all real values of x. The quadratic $ax^2+bx+c=0$ has its roots imaginary if $b^2-4ac < 0$.

Similarly the calculation of Sturmian functions will stop at the stage when any of them becomes a perfect square for it too will have an invariable sign for all real values of x.

3. In case any of the Sturmian functions $f_r(x)$ vanishes for $x=\alpha$, then for counting the number of changes of sign in the sequence

$$f(\alpha), f'(\alpha), f_1(\alpha), f_2(\alpha) \ldots,$$

we may regard the sign of $f_r(\alpha)$ either +ive or −ive because the function that precedes it and the one which follows it have opposite signs.

Working Rule.

1. Find the Sturm's functions as explained.

2. The last Sturm's function will be numerical in case $f(x)=0$ has no equal roots otherwise it will be some function of x.

3. The calculation of Sturm's functions should stop at the stage when any of them is either a perfect square or has all its roots imaginary.

4. Make a table as explained below :—

(i) In the first column write down

$$x, f, f', f_1, f_2, f_3 \ldots$$

In the first row write down x and the various values, that you may give to x.

(ii) In the various other columns write down the signs of the Sturm's functions corresponding to the value of x written at the top of that column.

(iii) At the bottom of each column write down the number of changes of signs in that particular column.

(iv) If we are to find only the number of real roots, we put $x=\infty$, and $-\infty$ and find out the difference of the changes of signs corresponding to these two values.

(v) If we want to find out the +ive and −ive roots, we put $x=\infty$, 0 and $-\infty$ and proceeding as above, we get the number of +ive roots which will lie between 0 and ∞ and −ive roots which lie between 0 and $-\infty$.

(vi) If we want to find the interval in which the roots lie, then for +ive roots we put $x=1, 2, 3, \ldots$ and for −ive roots we put $x=-1, -2, -3, \ldots$.

(viii) If corresponding to the two values of x say 2 and 3 the changes of sign in the Sturm's function be same *i.e.* their difference be zero, then no root will lie between 2 and 3.

Ex. 1. *Find the number and position of the real roots of the equation* $x^6-2x^2+3x-4=0$. **(Delhi Hon's 52)**

$$f(x)=x^6-2x^2+3x-4.$$

$$f'(x)=6x^5-4x+3.$$

Multiplying $f(x)$ by 6, and then dividing by $f'(x)$,

$$6x^5-4x+3)\;6x^6-12x^2+18x-24\;(x$$

$$\underline{6x^6-4x^2+3x}$$

$$-8x^2+15x-24.$$

Changing the sign of this remainder, we get

$$f_2(x)=8x^2-15x+24.$$

Now since b^2-4ac *i.e.* $(15)^2-4.8.24$ is −ive, we conclude that the roots of the Sturm's functions $f_2(x)=0$ are imaginary and hence we stop further calculations.

x	$-\infty$	0	∞	1	2	-1	-2
$f(x)$	+	−	+	−	+	−	+
$f'(x)$	−	+	+	+	+	+	−
$f_2(x)$	+	+	+	+	+	+	+
No. of changes of signs	2	1	0	1	0	1	2

From the above table by Sturm's theorem we have the following regarding the nature of the roots :—

1. There are only two real roots (from columns 2 and 4).

2. One of them is +ive (from columns 3 and 4) which lies between 1 and 2 (from columns 5 and 6).

3. One of them is −ive (from columns 2 and 3) which lies between −1 and −2 (from columns 7 and 8).

4. The equation being of sixth degree has only two real roots, therefore remaining four roots are imaginary.

Ex. 2. *Find the number of distinct real roots of the equation* $x^3-3x+1=0$ *and locate them.*

(Delhi Hon's 48, 51, 54)

$f(x)=x^3-3x+1,$

$f'(x)=3(x^2-1)$ or (x^2-1) on dividing by 3,

$f_2(x)=2x-1$ [after changing sign of the remainder].

$f_3(x)=3$ [after changing the sign]

x	$-\infty$	0	∞	1	2	-1	-2 2
$f(x)$	$-$	$+$	$+$	$-$	$+$	$+$	$-$
$f'(x)$	$+$	$-$	$+$	$+$ or $-$ any	$+$	$+$ or $-$ any	$+$
$f_2(x)$	$-$	$-$	$+$	$+$	$+$	$-$	$-$
$f_3(x)$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
No. of changes of signs	3	2	0	1	0	2	3

As in example 1 the above table shows that the given equation has all its roots real and distinct out of which one is $-$ive and the other two $+$ive. The $-$ive root lies in the interval $(-2, -1)$ and one of the $+$ive roots lies in the interval $(1, 2)$ and the other in the interval $(0, 1)$.

Ex. 3. *(a) Discuss the nature and position of the roots of the equation $x^4-12x^2+12x-3=0$ by means of Sturmian functions.* **(Agra 46)**

$f(x)=x^4-12x^2+12x-3$,

$f'(x)=4(x^3-6x+3)$ or x^3-6x+3 on dividing by 4.

$$\begin{array}{r|l}
x^3-6x+3) & \overline{x^4-12x^2+12x-3}(x \\
 & x^4-6x^2+3x \\
 & \overline{-6x^2+9x-3}
\end{array}$$

changing the sign and dividing by 3.

$$
\begin{array}{l}
f_2(x)=2x^2-3x+1)\overline{x^3-6x+3}(x+3 \\
\qquad\qquad\qquad\qquad\;\; 2 \\
\qquad\qquad\qquad\qquad\;\; \overline{2x^3-12x+6} \\
\qquad\qquad\qquad\qquad\;\; 2x^3-3x^2+x \\
\qquad\qquad\qquad\qquad\;\; \overline{3x^2-13x+6} \\
\qquad\qquad\qquad\qquad\;\; 2 \\
\qquad\qquad\qquad\qquad\;\; \overline{6x^2-26x+12} \\
\qquad\qquad\qquad\qquad\;\; 6x^2-9x\;+3 \\
\qquad\qquad\qquad\qquad\;\; \overline{\quad -17x+9} \text{ changing the sign.}
\end{array}
$$

$f_3(x)=17x-9.$

Now $f_4(x)$ will be numerical and we are concerned only with its sign, $f_2(x)=0$ gives $x=9/17$ *i.e.* slightly $> \frac{1}{2}$ say $\frac{1}{2}+K$ where K is small. This value of x when substituted in $f_2(x)$ makes it

$$2\left(\tfrac{1}{2}+K\right)^2-3\left(\tfrac{1}{2}+K\right)+1=2K^2-K$$

i.e. −ive for small values of K and hence $f_4(x)$ is +ive and numerical. It may be verified by actual calculation that $f_4(x)=8$.

x	$-\infty$	0	∞	1	2	3	−1	−2	−3	−4
$f(x)$	+	−	+	−	−	+	−	−	−	+
$f_1(x)$	−	+	+	−	−	+	+	+	−	−
$f_2(x)$	+	+	+	+ or −ive	+	+	+	+	+	+
$f_3(x)$	−	−	+	+	+	+	−	−	−	−
$f_4(x)$	+	+	+	+	+	+	+	+	+	+
No. of changes of signs	4	3	0	1	1	0	3	3	3	4

The above table shows that the given equation has all its roots real ; one of them is −ive lying in the interval

$(-4, -3)$. The other three are +ive lying in the interval $(2, 3)$ and two in the interval $(0, 1)$.

(b) *Find the number and position of real roots of*

$$f(x)=x^4-3x^3-2x^2-7x+3=0.$$

Proceeding as usual $f_1(x)=4x^3-9x^2-4x+7$

$$f_2(x)=43x^2-72x-69$$

$$f_3(x)=83x-191.$$

Since $f_4(x)$ will be numerical and we are concerned with its sign only, $f_3(x)=0$ gives $x=\frac{191}{83}$ which is slightly less than $2\frac{1}{3}$.

Let $x=\frac{7}{3}-h$ where h is small.

Putting for x in $f_2(x)$, we get

$$f_2(x)=43\left(\tfrac{7}{3}-h\right)^2-72\left(\tfrac{7}{3}-h\right)-69$$

$$=43h^2-128\tfrac{2}{3}h-2\tfrac{8}{9}.$$

Since h is small the sign of $f_2(x)$ is the same as that of the last term which is −ive and hence $f_4(x)=$ +ive.

	$-\infty$	$0'$	∞					-2
$f(x)$	+	+	+	+	+	+	−	+
$f'(x)$	−	+	+	−	−	+	−	−
$f_2(x)$	+	−	+	−	−	+	+	+
$f_3(x)$	−	−	+	−	−	+	−	−
$f_4(x)$	+	+	+	+	+	+	+	+
No of changes of signs	4	2	0	2	2	0	3	4

Above table shows that $f(x)=0$ has got all its roots real out of which two are +ive and two −ive. Both the +ive roots lie between 2 and 3, one −ive root lies between $(-1, 0)$ and the other between -2 and -1.

Ex. 4. *Find the nature of the roots of the equation*

$$x^4-5x^3+9x^2-7x+2=0. \qquad \textbf{(Agra 55)}$$

$$f(x)=x^4-5x^3+9x^2-7x+2$$

$$f'(x)=4x^3-15x^2+18x-7$$

$$f_2(x)=x^2-2x+1.$$

Now $f_2(x)$ divides $f'(x)$ without any remainder and as such we conclude that x^2-2x+1 *i.e.* $(x-1)^2$ is the H. C. F. of $f(x)$ and $f'(x)$. Hence $(x-1)^3$ must be a factor of $f(x)$ showing that $f(x)=0$ has three roots each equal to 1.

x	$-\infty$	0	∞	2
$f(x)$	+	+	+	0 ∴ 2 is a root
$f'(x)$	−	−	+	
$f_2(x)$	+	+	+	
No. of changes of signs	2	2	0	

From the above table, we find that there are only two distinct real roots which are +ive. Since $f(2)=0$, ∴ 2 is one of the roots and the other root is 1 which is a triple root. Thus the equation has all its roots real; three of them being each equal to 1 and the fourth equal to 2.

Ex. 5. *Show by the use of Sturm's theorem that the equation* $2x^6-18x^5+60x^4-120x^3-30x^2+18x-5=0$ *has four imaginary* *roots and that its +ive root lies in the interval (5, 6).*

Here proceeding as usual we find that

$$f_2(x)=5x^4+220x^2+1$$

and this expression is always +ive for all real values of x and we can divide by it taking $f_2(x)=1$. Now proceed as usual.

x	$-\infty$	0	∞	1	2	3	4	5	6	-1
$f(x)$	+	−	+	−	−	−	−	−	+	+
$f_1(x)$	−	+	+	−	−	−	−	+	+	
$f_2(x)$	+	+	+	+	+	+	+	+	+	+
Changes of sign	2	1	0	1	1	1	1	1	0	2

Above table shows that there are only two real roots and hence the remaining four roots are imaginary. The +ive root lies between 5 and 6 whereas the −ive root lies between 0 and −1.

Ex. 6. *Find by Sturm's method the number and location of the real roots of the equation*

$$x^5-10x^3+6x+1=0. \qquad \textbf{(Agra 44, 47)}$$

Here $f(x)=x^5-10x^3+6x+1$

$f'(x)=5x^4-30x^2+6$

$f_2(x)=20x^3-24x-5$

$f_3(x)=96x^2-5x-24$

$f_4(x)=43651x+10920.$

Clearly $f_5(x)$ will be numerical and we are concerned only with its sign. From $f_4(x)=0$ we get $x=-\frac{10920}{43651}$ *i.e.* slightly greater than $-\frac{1}{4}$. Let it be $-\frac{1}{4}+h$ where h is small.

$$\therefore\ f_3(-\tfrac{1}{4}+h)=96(-\tfrac{1}{4}+h)^2-5(-\tfrac{1}{4}+h)-24$$

$$=96h^2-53h-\tfrac{67}{4}.$$

For small values of $h, f_3(-\frac{1}{4}+h)$ is clearly $-$ive and hence $f_5(x)$ will be $+$ive

x	$-\infty$	0	∞	1	2	3	4	-1	-2	-3	-4
$f(x)$	$-$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$-$
$f_1(x)$	$+$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$+$
$f_2(x)$	$-$	$-$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
$f_3(x)$	$+$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$f_4(x)$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
$f_5(x)$	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
No of changes of signs	5	2	0	1	1	1	0	4	4	4	5

From the above table it is quite clear that the roots are all real, 3 $-$ive and 2 $+$ive, one in $(-4, -3)$, two in $(-1, 0)$, one in $(0, 1)$ and one in $(3, 4)$.

Ex. 7. *Use Sturm's method to show that the equation* $x^4-12x+7=0$ *has a root between 2 and 3.* **(Agra 43, 52, 59)**

Ex. 8. *Use Sturm's theorem to determine the number and position of the real roots of the equation*

(*a*) $x^4-6x^3+5x^2+14x-4=0.$ **(Agra 50)**

(*b*) $x^4-6x^3+10x^2-5x-22=0.$ **(Agra 33)**

(*c*) $x^4-8x^3+25x^2-36x+8=0.$ **(Delhi Hon's 54)**

(*d*) $x^4-4x^3+7x^2-6x-4=0.$ **(Delhi Hon's 43)**

Ans. (*b*) Two real $(4, 5)$ and $-$ive $(0, -1)$.

(*d*) Two imaginary, $(-1, 0)$, $(2, 3)$.

Ex. 9. *Apply Sturm's theorem to prove that the equation* $x^3-2x-5=0$ *has only one* $+$*ive real root lying between 2, and 3.*

(Agra 35)

Ex. 10. *Prove that the equation* $x^5-x-16=0$ *has two pairs of complex roots.* **(Delhi Hon's 50)**

$f_2(x)=x+20$ and $f_3(x)=-$ive one +ive root between 1 and 2.

Ex. 11. *Prove by Sturm's method that the equation* $x^4-6x^3-13x^2-12x+4=0$ *has two pairs of equal roots* $f_2(x)=(x-1)(x-2)$ *etc.* **(Agra 57)**

Roots are 1, 1 ; 2, 2.

Ex. 12. *Analyse the equation*

$$x^6-7x^5+15x^4-40x^2+48x-16=0.$$

$$f_3(x)=x^3-6x^2+12x-8=(x-2)^3 \text{ etc.}$$

Four roots are equal each equal, to 2 ; of the remaining two roots one lies in the interval (0, 1) and the other in $(-2, -1)$.

Ex. 13. *Calculate Sturm's functions for the quartic* $z^4+6Hz^2+4Gz+a^2I-3H^2=0.$ **(Agra 37, 49)**

$$f(z)=z^4+6Hz^2+4Gz+a^2I-3H^2.$$

$$f'(z)=z^3+3Hz+G \qquad \text{on dividing by (4).}$$

$$z^3+3Hz+G)z^4+6Hz^2+4Gz+a^2I-3H^2(z$$

$$\underline{z^4+3Hz^2+Gz}$$

$$3Hz^2+3Gz+a^2I-3H^2.$$

$$\therefore\ f_2(z)=-3Hz^2-3Gz-(a^2I-3H^2)\,)z^3+3Hz+G(-Hz+G$$

Multiply by a +ive quantity $3H^2$ $\quad 3H^2$

$$\overline{3H^2z^3+9H^3z+3H^2G}$$

$$3H^2z^3+3GHz^2+(a^2HI-3H^3)z$$

$$\overline{-3GHz^2-(a^2IH-12H^3)z+3H^2G}$$

$$-3GHz^2-3G^2z-(a^2I-3H^2)G$$

$$\overline{z\{3(G^2+4H^3)-a^2IH\}+a^2IG}$$

$$=za^2(2HI-3aJ)+a^2IG\ ;\ \because\ G^2+4H^3=a^2(HI-aJ)$$

Cancelling a^2 and changing the sign, we get

$$3H(2HI-3aJ)z^2+3G(HI-3aJ)$$

$$f_3(z)=-z(2HI-3aJ)-IG)\overline{-3Hz^2-3Gz-(a^2I-3H^2)}\quad($$

Multiplying by the +ive quantity $(2HI-3aJ)^2$,

$$-3H(2HI-3aJ)^2z^2-3G(2HI-3aJ)^2z-(a^2I-3H^2)(2HI-3aJ)^2$$

$$-3H(2HI-3aJ)^2z^2-3HIG(2HI-3aJ)z$$

$$-3Gz(2HI-3aJ)[HI-3aJ]-(a^2I-3H^2)(2HI-3aJ)^2$$

$$-3Gz(2HI-3aJ)[HI-3aJ]-3G^2I(HI-3aJ)]$$

$$3G^2I(HI-3aJ)-(a^2I-3H^2)(2HI-3aJ)^2$$

$$=3G^2HI^2-9aG^2IJ-(a^2I-3H^2)[4H^2I^2-12HIJa+9a^2J^2]$$

$$=3G^2HI^2-9aG^2IJ-4a^2H^2I^3+12H^4I^2+12a^3HI^2J$$

$$-36H^3IJa-9a^4IJ^2+27a^2H^2J^2$$

$$=3HI^2(G^2+4H^3)-9aIJ(G^2+4H^3)-4a^2I^3H^2)-4a^2I^3H^2-9a^4IJ^2$$
$$+12aH^3I^2J+27H^2a^2J^2.$$

Putting $G^2+4H^3=a^2HI-a^3J$

$$=(3HI^2-9aIJ)(a^2HI-a^3J)-4a^2I^3H^2-9a^4IJ^2$$
$$+12a^3HI^2J+27H^2a^2J^2$$

$$=3a^2H^2I^3-9a^3HI^2J-3a^3HI^2J+9a^4IJ^2$$
$$-4a^2I^3H^2-9a^4IJ^2+12a^3HI^2J+27H^2a^2J^2$$

$$=-a^2I^3H^2+27H^2a^2J^2.$$

Cancelling the +ive factor a^2H^2 and changing the sign we get $f_4(z)\equiv I^3-27J^2$ which is called the discriminant of the biquadratic. [See Ex. 1, P. 139].

Ex. 14. *Verify by means of Sturm's remainders, the conditions which must be fulfilled when the biquadratic*

$$ax^4+4bx^3+6cx^2+4dx+e=0$$

is a perfect square and prove in that case

$$a^3f(x)=\{(ax+b)^2+3H\}^2. \qquad \textbf{(Agra 39)}$$

[See Q. 4 P. 135 and Q. 12 P. 166].

Ex. 15. *Prove that when the biquadratic of the last example has a triple factor, it may be expressed in the form*

$$a^3f(x)=\{ax+b+\sqrt{(-H)}^3\}\{ax+b-3\sqrt{(-H)}\}.$$

[See Q. 11 P. 164].

Ex. 16. *Find Sturm's functions for the cubic*

$$z^3+3Hz+G=0,$$

$f(z)=z^3+3Hz+G$, $f'(z)=z^2+H$, $f_2(z)=-2Hz-G$,
$f_3(z)=-(G^2+4H^3)$.

§ 4. Condition that all the roots of a given equation be real and distinct.

1. In case the equation has all its roots distinct, then the sequence of Sturm's functions must in general consist of $(n+1)$ functions *i.e.*, the given function, its derivative and $(n-1)$ Sturm's remainders.

2. In case the equation has all its roots real, then the difference of the number of changes of sign when x is put ∞ in the sequence over the number when x is put $-\infty$ should be n. In other words it means that when x is put $+\infty$ no change of sign occurs *i.e.* the number of changes of sign be zero and when $x=-\infty$ in the $(n+1)$ functions, they should be alternately +ive and -ive so that there be n change of sign and the difference of these changes of sign be $n-0=n$, the number of real and distinct roots.

The above conditions are satisfied if we say that the leading coefficients of all the Sturm's functions be +ive. (We always take the leading coefficient of a given equation to be +ive.)

Ex. 1. *Use Sturm's theorem to show that the equation* $z^3+3Hz+G=0$ *has all the three roots real and distinct if and only if* $G^2+4H^3<0$. **(Agra 48)**

The leading coefficient of all the Sturm's functions is +ive and from Ex. 16, we get the conditions as

$$-H \text{ +ive} \quad \text{and} \quad -(G^2+4H^3) \text{ +ive}.$$

i.e. $$H \text{ -ive and } G^2+4H^3 \text{ -ive}.$$

But G^2+4H^3 can be -ive only if H is -ive and hence the former condition is implied in the latter. Hence G^2+4H^3 is $-$ive *i.e.* < 0 if all the roots of the given cubic are real and distinct.

§ 5. Nature of the roots of Biqudratic.

By the help of Sturm's theorem, discuss the nature of the roots of the equation $ax^4+4bx^3+6cx^2+4dx+e$.

The above equation can be reduced to the form

$$f(z)=z^4+6Hz^2+4Gz+a^2I-3H^2=0,$$

where $z=ax+b$ and the nature of the roots of the given equation and of the transformed equation is same. From Ex. 13. P. 193, the Sturm's functions are

$$f(z)=z^4+6Hz^2+4Gz+a^2I-3H^2$$
$$f'(z)=z^3+3Hz+G$$
$$f_2(z)=-3Hz^2-3Gz-(a^2I-3H^2)$$
$$f_3(z)=-z(2HI-3aJ)-IG$$
$$f_4(z)=I^3-27J^2=\Delta \text{ the discriminant.}$$

1. All roots real and distinct—The leading coefficient of Sturm's function should be $+$ive.

$\therefore$ H $-$ive, $2HI-3aJ$ $-$ive, I^3-27J^2 $+$ive.

2. All roots imaginary—

I^3-27J^2 $+$ive, and either H $+$ive or $2HI-3aJ$ $+$ive.

In this case corresponding to $z=\infty$, the number of changes of sign in the above squence will be

$$z=\infty \quad ++-+ \text{ or } -+2.$$
$$z=-\infty \quad +--+ \text{ or } -+2.$$

The difference corresponding to the values ∞ and $-\infty$ of z is zero and hence the equation has no real roots. Thus all its roots are imaginary.

3. Two roots real and two imaginary—

I^3-27J^2 – ive : In this case we shall find that the difference of the changes of signs corresponding to $z=\infty$ and $-\infty$ in the above sequence is always 2. We may give H and $2HI-3aJ$ any sign we like. Thus the equation has two real and two imaginary roots.

4. Two equal roots—In this case clearly $I^3-27J^2=0$ and then $f_3(z)=0$ will give us the H. C. F. of $f(z)$ and $f'(z)$, and thus proving the existense of two equal roots.

5. Three equal roots or two pairs of equal roots : In this case the H. C. F. should be of 2nd degree and also a perfect square or composed of two unequal factors according as three roots are equal or two pairs of equal roots exist. Hence $f_3(z)$ should vanish identically which will happen when either (1) $I=0$ and $J=0$

or (2) $G=0$ and $2HI-3aJ=0$.

When $I=0$, $J=0$, we get $G^2+4H^3=a^2(HI-aJ)=0$ and $f_2(z)=0$ becomes $3Hz^2+3Gz-3H^2=0$ and its discriminant is $9(G^2+4H^3)$ which is zero and hence its roots are equal *i. e.* it is a perfect square. Since the H. C. F. is a perfect square $f(z)=0$ has three equal roots.

When $G=0$ and $2HI-3aJ=0$, then $f_2(z)$ is the H. C. F. but is not a perfect square and hence the equation will have a pair of equal roots.

Note—Now $G^2+4H^3=a^2HI-a^3J$. Putting $G=0$ and $J=\frac{2HI}{3a}$, we get $4H^3=a^2HI-a^3.\frac{2HI}{3a}$ or $a^2I=12H^2$ and this is same as the condition we had found in Ex. 12 P. 166.

6. All roots equal—If $I=0$, $H=0$ and $G=0$, then $f_3(z)$ vanishes identically and the H. C. F. comes out to be $f'(z)$ *i. e.* z^3 showing that all the roots are equal.

CHAPTER VIII

SOLUTION OF NUMERICAL EQUATIONS

§ **1. Algebraical Equations** :—Those equations in which the various coefficients are not numerical numbers, but algebraical quantities are called algebraical equations. The roots of such equations are determined in terms of the coefficients, just as the quadratic $ax^2+bx+c=0$ has for its roots $\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$. The algebraic solutions of cubic and biquadratic equations have already been given in Chapter V.

Numerical Equations :—In the case of numerical equations, the coefficients are given numbers and we can find either the exact or approximate value of a commensurable root of the equation by the methods to follow.

Commensurable and incommensurable roots :—A real root of a given numerical equation is said to be commensurable when it is an integer, or a rational function or terminating or repeating decimals which are reducible to fractions. The incommensurable roots of an equation consist of interminable decimals. For examples 4, 7, $\frac{1}{5}$, $\frac{1}{4}$ are commensurable whereas $\sqrt{3}$, $\sqrt{13}$, e, π are incommensurable.

§ **2. Theorem** :—*An equation with integral coefficients and having unity for the coefficient of the first term, cannot have a commensurable root which is not an integer. The integer root of such an equation is a factor of the absolute term.* **(Pb. 52)**

Let the equation be $x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=0$ where all the a's are integers. If possible, let us suppose that it has a root $\frac{p}{q}$ a fraction in its lowest terms.

$$\therefore\quad \left(\frac{p}{q}\right)^n+a_1\left(\frac{p}{q}\right)^{n-1}+a_2\left(\frac{p}{q}\right)^{n-2}+\ldots+a_n=0.$$

Now multiplying both sides by q^{n-1}, we get

$$-\frac{p^n}{q}=a_1p^{n-1}+a_2p^{n-2}q+\ldots+a_nq^{n-1}.$$

Now since p and q have no common factor, hence L.H.S. *i.e.* $\frac{p^n}{q}$ is a fraction in its lowest terms whereas in the R. H. S. all the terms are integers. But it is impossible for a fraction to be equal to an integer and hence our supposition is wrong.

Hence if the equation has commensurable roots, they must be integers. Or, the real roots of the equation of the type as given are either integers or incommensurable.

Note :—In case the coefficient of highest term be not unity but say a_0 or other coefficients be not integral, then we can reduce this equation to another by multiplying the roots of the given equation by a_0 or by the L. C. M. of the denominators of the various fractional coefficients or a number less than that so that all the coefficients of the transformed equation are integers and the coefficients of leading term unity [§ 4 P. 37]. The roots of this transformed equation will be integers only. Having found them we can find the roots of the original equation as $y=a_0x$ or Kx.

From the given equation we find that product of all the roots is numerically equal to a_n which is an integer and the roots too are integers. Hence we conclude that each root must be a factor of a_n.

Thus for finding the integer roots of the equation we should try a factor K of the absolute term a_n for the root of $f(x)\equiv 0$. If K is a root then $f(K)=0$ *i.e.* $(x-K)$ divides $f(x)=0$ exactly. We shall give below another method of finding out the integer roots of a given equation which is known as Newton's Method of Divisors.

§ 3. Newton's Method of Divisors for obtaining integral roots of a given equation.

(Delhi Hon's 48, Pb. 32, 36, 39, 51)

Let h be an integer root of the equation

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0$$

where all the a's are integers. Again suppose that when $f(x)$ is divided by $(x-h)$, the quotient is $\phi(x)$ where $\phi(x)\equiv b_0x^{n-1}+b_1x^{n-2}+b_2x^{n-3}+\ldots b_{n-2}x+b_{n-1}$ where all the b's must be integers. Then we have

$$\begin{aligned} f(x)&\equiv(x-h)\,\phi(x)\\ &\equiv(x-h)\,(b_0x^{n-1}+b_1x^{n-2}+b_2x^{n-3}+\ldots+b_{n-2}x+b_{n-1}). \end{aligned}$$

Comparing the coefficients of like powers of x, we have the following relations :

$$\left.\begin{array}{l} a_0=b_0,\ a_1=b_1-hb_0,\ a_2=b_2-hb_1\ldots\ldots \\ a_r=b_r-hb_{r-1},\ a_{n-2}=b_{n-2}-hb_{n-3} \\ a_{n-1}=b_{n-1}-hb_{n-2} \text{ and } a_n=-hb_{n-1} \end{array}\right\}\ldots\ldots\ldots(A)$$

Rewriting the above relations in reverse order, we have

$$\left.\begin{array}{l} \dfrac{a_n}{h}=-b_{n-1},\quad \dfrac{-b_{n-1}+a_{n-1}}{h}=-b_{n-2} \\ \dfrac{-b_{n-2}+a_{n-2}}{h}=-b_{n-3}\ldots\ldots\ldots\dfrac{-b_r+a_r}{h}=b_{r-1} \\ \ldots\ldots\dfrac{-b_2+a_2}{h}=-b_1,\quad \dfrac{-b_1+a_1}{h}=-b_0 \text{ and } -b_0+a_0=0 \end{array}\right\}\ldots(B)$$

The relation B suggests the following method for testing any number h for the root of the given equation.

Divide a_n by h and the quotient is $-b_{n-1}$. Add the quotient to a_{n-1} and again divide by h and the quotient is $-b_{n-2}$. Add the quotient to a_{n-2} and again divide by h and the quotient is $-b_{n-3}$. Continue the process and you will get the last quotient as $-b_0$ which when added to a_0 must give zero in case h is a root. We know that all the b's *i.e.* various quotients are integer and hence if at any stage of division we find that any of the quotients is not an integer we should conclude that h is not the root and the process must stop.

In the light of relations A and B the above procedure can be exhibited in a simple way as follows :—

$$\begin{array}{c|cccccc}
\underline{h} & a_n & a_{n-1} & a_{n-2}\ldots\ldots a_2 & & a_1 & a_0 \\
 & & -b_{n-1} & -b_{n-2}\ldots\ldots -b_2 & & -b_1 & -b_0 \\
\hline
 & & -hb_{n-2} & -hb_{n-3}\ldots & -bh_1 & -hb_0 & 0
\end{array}$$

Procedure :— The first line denotes the coefficients of the given equation written in **reverse order** (supply the missing coefficients by zero.) The second line denotes the coefficients of the quotients written in **reverse order with sign changed.** The third line gives the number obtained by adding the corresponding numbers of 1st and 2nd lines and all of them are divisible by h, If at any stage any number of 3rd row is not divisible by h, the process should stop and we should conclude that h is not a root. In case h be a root the last number in the 3rd row should be zero. Divide a_n by h and write the quotient in 2nd line. Add it to the corresponding number of 1st line and we get the 1st number of 3rd line. Divide it by h

and write in 2nd line. Adding it to the corresponding number of 1st we get the 2nd number of 3rd line and so on.

Note—The above method gives only the integer roots of the equation,

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0.$$

Also we know that if the coefficient of the leading term be unity and all other coefficients integers then the integer roots are the only commensurable roots. Hence in order to find all the commensurable roots of the given equation we should reduce the equation to the form in which the coefficients of leading terms be unity and the integer roots (as it will have only integer roots) of the transformed equation may be found. Having found these integer roots, we can find all the commensurable roots of the given equation :—

Note :—Having found that any integer is a root of the given equation we, should try other integers to be a root of the depressed equation whose coefficients will be the **elements of 2nd line in reverse order with sign changed.**

§ 4. Restricting the number of trial divisors in Newton's method. (Pb. 32, 36, 39, 51)

We have seen that in an equation whose coefficients are integers and leading coefficient unity all the integer roots are the factors of the absolute term a_n. In case a_n be large we shall have to test a number of different factors of a_n for the roots of the given equation. The following methods will be of considerable help :—

1. If h be a root of the given equation, then

$f(x)=(x-h)\ \phi(x)$ where $\phi(x)$ is the quotient.

$\therefore\ f(1)=(1-h)\ \phi(1)$ or $-(h-1)\phi(1)$

$f(-1)=(-1-h)\phi(-1)$ or $-(h+1)\phi(-1)$.

Above shows that $h-1$ must divide $f(1)$ and $(h+1)$ must divide $f(-1)$ in case h be a root which will be a factor of a_n. Hence all those trial divisors *i. e.* factors of a_n which when increased by one fail to divide $f(-1)$ should be rejected, and all those which when decreased by one fail to divide $f(1)$ should be rejected.

2. Find out the upper and lower limits of the real roots of the given equation and all those factors of a_n which fall beyond these limits should be rejected.

3. Some of the trial divisors can be rejected by performing the Newton's Method mentally as you will find that just in the begining the elements of the 3rd row in Newton's method will not be divisible by h, the trial divisor.

4. In case $f(1)=0$ or $f(-1)=0$, then 1 or -1 is a root. and we can depress the equation by one dimension.

5. Repeated roots—In case h be a double root, then h^2 must be a factor of last coefficient and h that of last but one. If h be a triple, then h^3, h^2 and h should be factors of last, last but one and last but two coefficients respectively of the given equation.

Ex. 1. *The roots of the equation*

$$x^4-x^3-30x^2-76x-56=0$$

lie between -6 *and* 12. *Find them.* **(Pb., 39)**

The possible factors of 56 that lie between -6 and 12 are ± 2, ± 4, 7, 8.

Also $f(1)=-162$ and $f(-1)=-8$.

Reject all those divisors which when increased by one fail to divide $f(-1)$, thus rejecting $2, \pm 4, 8$ and we are left with -2 and 7. Reject all those divisors which when decreased by one fail to divide $f(1)$. -2 and 7 are not rejected and we proceed to find whether they are the roots or not.

$$\begin{array}{l|rrrrr}
-2 & -56 & -76 & -30 & -1 & 1 \\
 & & -28 & 24 & 3 & -1 \\
\hline
 & & -48 & -6 & 2 & 0
\end{array}$$

Thus -2 is a root and the depressed equation is $x^3-3x^2-24x-28=0$ as the 2nd line gives the coefficients of the depressed equation in reverse order with sign changed.

Let us try the trial divisor -7 on this depressed equation.

$$\begin{array}{l|rrrr}
7 & -28 & -24 & -3 & 1 \\
 & & -4 & -4 & -1 \\
\hline
 & & -28 & -7 & 0
\end{array}$$

Thus 7 is also a root and the depressed equation is $x^2+4x+4=0$ whose roots are clearly $-2, -2$.

Thus the commensurable roots (which will be integers only) are $7, -2, -2, -2$.

Note.—Since -2 is a triple root it may be seen $(-2)^3$, $(-2)^2$ and (-2) are the factors of last, last but one, last but two coefficients respectively.

Had we tried any number other than -2 or 7, we would have found that the element of 3rd row will not be divisible by that number showing that that number is not a root.

Ex. 2. *Find the commensurable and multiple roots of the equation* $x^5-4x^4-2x^3+14x^2-3x+18=0$. **(Pb. 45)**

The divisors to be tested are $\pm(9, 6, 3, 2)$.

Again $f(1)=24$, $f(-1)=32$.

By the rule explained in last example ± 9, ± 6 are rejected. Because 9 when decreased by 1 divides $f(1)$ but when increased by one does not divide $f(-1)$. Hence we are left with ± 3, ± 2.

$$\begin{array}{r|rrrrrr} 3 & 18 & -3 & 14 & -2 & -4 & 1 \\ & & 6 & 1 & 5 & 1 & -1 \\ \hline & & 3 & 15 & 3 & -3 & 0 \end{array}$$

Hence 3 is a root. Again 3^2 and 3 are present in last and last but one coefficients respectively and as such there is a possibility of 3 being a repeated root. We try the trial divisor 3 on the depressed equation.

$$\begin{array}{r|rrrrr} 3 & -6 & -1 & -5 & -1 & 1 \\ & & -2 & -1 & -2 & -1 \\ \hline & & -3 & -6 & -3 & 0 \end{array}$$

Thus 3 is a double root. Now we shall try the trial divisor 2 on the depressed equation.

$$\begin{array}{r|rrrr} 2 & 2 & 1 & 2 & 1 \\ & 1 & 1 & & \\ \hline & 2 & 3 & & \end{array}$$

Stop since 3 is not divisible by 2.

Let us try -2.

$$\begin{array}{r|rrrr} -2 & 2 & 1 & 2 & 1 \\ & & -1 & 0 & -1 \\ \hline & & 0 & 2 & 0 \end{array}$$

Thus -2 is a root and the depressed quadratic is

$$x^2+1=0, \quad \therefore \quad x=\pm i.$$

Hence commensurable roots are 3, 3, -2.

Ex. 3. *Find by Newton's method of divisors the integral roots of the equation* $6x^4-25x^3+26x^2+4x-8=0$ *and hence solve it.* **(Pb. 46)**

In order to make the coefficient of leading term unity we have to multiply the roots by 6 which will make the absolute term of transformed equation quite big. We first find the equation whose roots are the reciprocals of the roots of the given equation and that equation is

$$8y^4-4y^3-26y^2+25y-6=0$$

or
$$y^4-\frac{1}{2}y^3-\frac{13}{4}y^2+\frac{25}{8}y-\frac{3}{4}=0.$$

In order to make all the coefficients integers we should multiply the roots of the above equation by 2 and the transformed equation is

$$z^4-z^3-13z^2+25z-12=0. \text{ where } z=2y=2.\frac{1}{x},$$

$f(1)=0$ and hence 1 is a root and depressed equation is

$$\begin{array}{r|rrrrr}
1 & -12 & 25 & -13 & -1 & 1 \\
 & & -12 & 13 & 0 & -1 \\
\hline
 & & 13 & 0 & -1 & 0
\end{array}$$

$z^3-13z+12=0$. Again 1 is clearly a root of this equation and the depressed equation is

$$\begin{array}{r|rrrr}
1 & 12 & -13 & 0 & 1 \\
 & & 12 & -1 & -1 \\
\hline
 & & -1 & -1 & 0
\end{array}$$

$$z^2+z-12=0 \text{ or } (z+4)(z-3)=9, \quad \therefore \quad z=3,\ -4.$$

Hence $z=1.\ 1,\ 3,\ -4$, and $x=\dfrac{2}{z}$.

$$\therefore \quad x=2,\ 2,\ \frac{2}{3},\ \frac{-1}{2}.$$

(b) *Find the commensurable and multiple roots of*

$$f(x)=9x^4-12x^3-71x^2-40x+16=0.$$

In order to make the coefficient of leading term unity we multiply the roots of the given equation by 3 and the resulting equation is

$$\phi(y)=y^4-4y^3-71y^2-120y+144=0,$$

where $y=3x$.

Proceeding as usual the only number to be tried for roots are 2 and -4. 2 can be shown not to be a root whereas -4 is a double root as $(-4)^2$ and -4 are the factors of the last and last but one coefficient.

$$\therefore \quad \phi(y)=(y+4)^2(y^2-12y+9)=0,$$

$$\therefore \quad y=-4,\ -4,\ 3(2\pm\sqrt{3}).$$

$$x=\frac{y}{3}=-\frac{4}{3},\ -\frac{4}{3} \text{ and } (2\pm\sqrt{3}).$$

Hence the multiple commensurable root is $-\frac{4}{3}$.

Ex. 4. *Find the commensurable roots of the equation*

$$x^5-29x^4-31x^3+31x^2-32x+60=0.$$

(Agra 60, Pb. 51, 55)

$f(1)=0$ and hence 1 is a root and the depressed equation as above is $\phi(x)=x^4-28x^3-59x^2-28x-60=0$

$$\phi(1)=-174,\ \phi(-1)=-62.$$

The divisors to be tested are $\pm(2, 3, 4, 5, 6, 10, 12, 15, 30, 60)$.

By applying the principle explained in Ex. 1, we are left with divisors -2, 30..

$$\begin{array}{r|rrrrr} 30 & -60 & -28 & -59 & -28 & 1 \\ & & -2 & -1 & -2 & -1 \\ \hline & & -30 & -60 & -30 & 0 \end{array}$$

Thus 30 is a root. Again try -2 on depressed equation.

$$\begin{array}{r|rrrr} -2 & 2 & 1 & 2 & 1 \\ & & -1 & 0 & -1 \\ \hline & & 0 & 2 & 0 \end{array}$$

Thus -2 is root and the depressed equation is

$$x^2+1=0, \quad \therefore \quad x=\pm i.$$

Hence the roots are 1, 30, -2, $\pm i$.

Exercise

1. Find the integral roots of the equation

$$x^5-23x^4+160x^3-281x^2-257x-440=0$$

being given that the limits of such roots are -1, 21.

(Pb. 58) Ans. 5, 8, 11.

2. Find the integral roots of the equation

$$3x^4-23x^3+35x^2+31x-30=0$$

having been given that the limits of such roots are -2, 6. **Ans.** -1, 3, 5.

Multiply the roots by 3 and the limits of the roots of new equation are -6, 18.

3. Find all the commensurable roots of

$$2x^3-31x^2+112x+64=0. \qquad \textbf{Ans. } \frac{-1}{2}, 8, 8$$

4. Solve the equation by Newton's method of divisors

$$2x^6-15x^5+32x^4+23x^3-186x^2+260x-120=0$$

being given that the roots lie between 3 and -3.

Ans. $\frac{3}{2}$; 2, 2, 2, $\pm\sqrt{5}$

5. The following equation has only two different roots. Find them :—

$$x^5-13x^4+67x^3-171x^2+216x-108=0.$$

Ans. 3, 3, 3, 2, 2

§ 5. Horner's Method :—

By the above method which is due to Horner, we can very conveniently find both the commensurable and incommensurable roots of a given equation $f(x)=0$. We first find the integer part of the roots and then proceed to find the decimal parts figure by figure. If the roots be commensurable then we can find it exactly and in case of incommensurable roots the value can be determined to any number of places of decimal which are obtained in succession.

We shall give below the method of finding the positive roots of $f(x)=0$. In order to find the $-$ive roots of $f(-x)=0$ we shall find the $+$ive root of $f(-x)=0$.

Let us consider an equation

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0 \quad .. \quad (1)$$

and suppose that it has a root say α whose integral part is p *i. e.* it has a single root in the interval $(p, p+1)$ where p is a $+$ive integer. We have stated here that there is a single root in the interval $(p, p+1)$ for if it has two roots in this interval, then those roots will be nearly equal and this case we shall consider later on. Let the root α in the interval $(p, p+1)$ be denoted by $p{\cdot}qrs\ldots\ldots$ where $q, r, s\ldots\ldots$ are the figures of the decimal part of the root.

1st step. Diminish the roots of (1) by p as explained in § 5, P. 42 and let the transformed equation be

$$b_0x^n+b_1x^{n-1}+b_2x^{n-2}+\ldots\ldots$$
$$+b_{n-1}x+b_n=0 \text{ where } b_0=a_0\ldots \quad \ldots(2)$$

The equation (2) has again a single root between 0 and 1 *i. e.* a root $0{\cdot}qrs\ldots\ldots$ In order to find the value of q, we multiply the roots of (2) by 10 [§ 4, P. 39] *i. e.* we multiply the 2nd, 3rd, 4th...terms by 10, 10^2, 10^3....respectively or we affix one zero, two zeroes, three zeroes with 2nd, 3rd, 4th... coefficients respectively and the transformed equation thus obtained is

$$\phi(x)\equiv b_0x^n+10b_1x^{n-1}+10^2b_2x^{n-2}+\ldots\ldots$$
$$+10^{n-1}b_{n-1}x+10^nb_n=0 \;\ldots\;\;\ldots\;\;\ldots(3)$$

Now the roots of (3) are the roots of (2) multiplied by 10, *i. e.* it has a single root between 0 and 10, the root of (2) is $\cdot qrs\ldots.$ and hence the single root of (3) is $q\cdot rs\ldots.$ Now diminish the roots of (3) by q which is the greatest integer of the root of $\phi(x)=0$.

Important Note—

Now we diminish the roots (3) by q and we obtain the transformed equation as

$$c_0x^n+c_1x^{n-1}+c_2x^{n-2}+\ldots.+c_{n-1}x+c_n=0\ldots\;\;\ldots(4)$$

It should be clearly noted that the sign of the last term of (4) is the same as that of the last term of (3).

Why ?—Equation (3) has single root between **0 and 10** and the greatest integer that we have found is q *i. e.* it has no root between **0 and q** and has a root between q and $q+1$.

Now $\phi(0)$=last term in the equation (3) and whose roots when diminished by q give the transformed equation whose last term will be nothing but the value of $\phi(q)$. [See P. 46]

Now if $\phi(0)$ and $\phi(q)$ be of opposite signs, then naturally a root of $\phi(x)=0$ will lie between 0 and q which contradicts that q is the greatest integer of (3). **Hence we conclude that if we are diminishing the roots by the greatest integer then the sign of the last term of the equation whose roots are being diminished and that of the transformed equation should be same.**

Effect of diminishing by a number greater than the greatest integer of the root.

The sign of the last term changes when we diminish the roots of (3) by a number greater than the greatest integer of the root of (3). For example, if we diminish the roots by $(q+1)$ which is greater than q, the greatest integer of the root of (4), then naturally $\phi(q)$ and $\phi(q+1)$ *i. e.* the last terms of the transformed equation when the roots of (3) are diminished by q and $(q+1)$ will be of opposite signs as a root of (3) lies between q and $q+1$. Now $\phi(q)$ has the same sign as that of the last term of (3) and as such $\phi(q+1)$ will have opposite sign.

Similarly if we diminish the roots by $q+2$, $q+3$....... then also the sign of the last term will change. **Hence we conclude that if at any stage of diminishing we find that the sign of the last term of transformed equation is of opposite sign to that of the last term of the equation whose roots are being diminished, we should feel that we have diminished the roots by an integer which is greater than the greatest integer of the root of the given equation.**

Note :—In the case of equal roots say 2·347 ; 2·604... when we diminish the roots by (6) we shall find that the sign of the last term will change as we shall cross the integer (3) which happens to be greatest integer for one of the nearly equal roots. The change of sign of last term wil happen only in the case of nearly equal roots.

It will provide a good check in the calculations.

Effect of diminishing the roots by a number less than the greatest integer of the roots of the equation.

The roots of (4) will now be ·*rs*... *i. e.* it will lie between **o and 1.** We again multiply the roots of (4) by 10 and the roots of the transformed equation will lie between **0 and 10** and the greatest possible next integer can be 9 only.

Now suppose you diminish the roots of (3) by a number less than the greatest integer of the root of (3) say by $(q-1)$; then the root of (4) will be $1{\cdot}rs\ldots$ *i. e.* it will lie between 1 and 2. We again multiply the roots of the transformed equation by 10 and the roots of the new equation will lie between 10 and 20 and as such the greatest integer of the root will be somewhere between 10 and 20 *i. e.* a number greater than 10.

Similarly if you diminish the roots of (3) by $(q-2)$, then the roots of transformed equation will be $2.rs\ldots$ *i.e.* it will lie between 2 and 3. We again multiply the roots of the transformed equation by 10 and the roots of the new equation will lie between 20 and 30 and as such the greatest integer of the roots will be somewhere between 20 and 30.

Hence we conclude that if at any stage of diminishing the number suggested by which the roots are to be diminished is greater than 10, we should feel that at the previous stage we had diminished the roots by a number which was less than the greatest integer of the root of that equation.

Newton's method of approximation :—

In the above process after we have diminished the roots 2 or 3 times and each time multiplying the roots by 10, it will be difficult to find the greatest integer by which the next diminishing is to be done. The following theorem gives an easy method for finding the digit by which the diminishing should be done after two, three stages (or even earlier).

After two or three transformations the next greatest integer of the root is obtained by dividing the last coefficient by the last but one.

Suppose a root of $f(x)=0$ differs from α by a small quantity h; then h is approximately equal to $\dfrac{-f(\alpha)}{f'(\alpha)}$.

Since $\alpha+h$ is a root, $\therefore\ f(\alpha+h)=0$.

We know that $f(\alpha+h)=f(\alpha)+hf'(\alpha)+\dfrac{h^2}{2!}f''(\alpha)+\ldots=0$.

Neglecting higher powers of h, we get

$$f(\alpha)+hf'(\alpha)=0 \quad \text{or} \quad h=\frac{-f(\alpha)}{f'(\alpha)}.$$

Now when we have diminished the roots by α, then $f(\alpha)$ is the last term of the transformed equation and $f'(\alpha)$ is the coefficient of last but one term. Hence the next number h is obtained by dividing the last coefficient by the last but one.

Again since h is +ive and we are dealing with +ive roots, we conclude that $f(\alpha)$ and $f'(\alpha)$ should be of opposite sign *i. e.* the last two coefficients of the transformed equation must be of opposite signs before we find the next integer by the above process.

The last but one coefficient in known as **trial divisor** and the above method is known as the **principal of trial divisor.**

Conclusions and working rule :—

(1) Try to find the integral part of the roots by the principle that if $f(a)$ and $f(b)$ be of opposite signs then at least a real root of $f(x)=0$ will lie between a and b. Choose a and b two consecutive integers.

(2) Diminish the roots by a and the roots of the transformed equation will then lie between 0 and 1 because a

and b are consecutive. Multiply the roots by 10 by annexing zeros, and again find the greatest integer of the root of the equation as in (1) and diminish again the root by this number. Continue this process.

(3) After two or three stages of diminishing apply the principle of trial divisor *i.e.* divide the last coefficient by the last but one coefficient to get the next integer by which the roots are to be diminished and these two coefficients should be of opposite signs.

(4) The sign of the last term should be maintained throughout after the first diminishing. In case it changes it should give you an indication that the diminishing is done by a number greater than the greatest integer of the root whose roots have been diminished. The sign of the last term will change in the case of nearly equal roots.

(5) If at any stage the greatest integer of the root comes out to be greater than 10, it should give you an indication that the diminishing is done by a number less than the greatest integer of the root whose roots have been diminished.

(6) **While diminishing the roots supply** the missing terms by zero coefficient. We shall illustrate all these points, checks, cautions in the following examples.

(7) **Important :**—It may happen that at any stage the trial divisor suggests the next integer to be zero. In such a case we should again multiply the roots by 10 by annexing zeros and write zero in decimal place of the root.

Ex. 1. *Find the +ive root of the equation*

$$x^3+x^2+x-100=0,$$

correct to four places of decimal. **(Agra 48, Delhi Hon's 48)**

$$f(x)=x^3+x^2+x-100=0 \quad \ldots \quad \ldots \quad \ldots(1)$$

(1) $f(4)=+$ive, $f(5)=-$ive and hence the +ive root lies between 4 and 5. Let the root be $4{\cdot}qrst\ldots\ldots$

(2) Diminish the roots of the given equation by 4 and we get the traosformed equation as

$$x^3+13x^2+57x-16=0 \quad \dots \quad \dots \quad \dots (2)$$

(3) The root of (2) lies between 0 and 1. Multiply its roots by 10 by annexing one zero, two zeros and three zeros with the 2nd, 3rd and 4th coefficients respectively, we get the transformed equation

$$\phi\,(x)=x^3+130x^2+5700x-16000=0 \quad \dots \quad \dots (3)$$

(4) The roots of (3) lie between 0 and 10. Clearly ϕ (2)=+ive and ϕ (3)=−ive and hence the greatest integer of the roots of (3) is 2 as is evident otherwise too by the principle of trial divisor ; if we divide 16000 by 5700 the number suggested is 2.

(5) Diminish the roots of (3) by 2 and we get the transformed equation as $x^3+136x^2+6332x-4072=0 \dots\dots (4)$ It is to be noted that the sign of the last term in (3) and (4) is same and the last and last but one coefficients have opposite signs.

Wrong Steps.

(a) Suppose we diminish the roots of (3) by 3 which is greater than 2, the greatest integer of the roots of (3). We find the transformed equation as

$$x^3+139x^2+6507x+2297=0.$$

We find that the last term does not maintain the same sign as that of (3) indicating that we have crossed the greatest integer. Also the last and last but one coefficients are not of opposite signs.

(b) Suppose we diminish the roots of (3) by 1, a number less than the greatest integer of the root of (3) ; then the transformed equation is

$$x^3+133x^2+5963x-10169=0.$$

Multiplying its roots by 10, we get

$$x^4+1330x^2+59630x-10169000=0$$

The trial divisor suggests a number greater than 10 by which the roots should now be diminished indicating that we have chosen for diminishing a number less than the greatest integer of the root of (3).

Right process continued.

(6) The roots of (4) lie between 0 and 1. Multiply its roots by 10 by annexing zeros and the transformed equation is $x^3+1360x^2+623200x-4072000=0$ ... ...(5)

(7) The roots of (5) lie between 0 and 10 and the trial divisor suggests the next number to be 6. $4072000 \div 623200$ suggests 6.

(8) Diminish the roots of (5) by 6 and we get the transformed equation as

$$x^3+1378x^2+639628x-283624=0 \quad \ldots \quad \ldots (6)$$

It is to be noted that the sign of the last term in (6) is the same as that of the last term of (5) and also the last and last but one coefficients in (6) are of opposite signs.

(9) The roots of (6) lie between 0 and 1. Multiply its roots by 10 by annexing zeros and the transformed equation is

$$x^3+13780x^2+63962800x-283624000=0 \ldots \quad \ldots (7)$$

(10) The roots of (7) lie between 0 and 10 and the trial divisor suggests the number 4.

(11) Diminish the roots of (7) by 4 and we get the transformed equation as

$$x^3+13792x^2+64073088x-27552256=0 \ldots \quad \ldots (8)$$

(12) The roots of (8) lie between 0 and 1. Multiply the roots by 10 by annexing zeros and the transformed equation is

$$x^3+137920x^2-6407308800x-27552256000=0 \ldots\ldots (9)$$

(13) The roots of (9) lie between 0 and 10 and the principle of trial divisor suggests the next figure to be 4 which is the fourth place of decimal and we need not carry on further process. Hence the +ive root of the given equation correct to four places of decimal is 4·2644.

Will you be prepared to find out the root of the above equation correct to nine places of decimal by proceeding as above ? I know you will not be ready. We will give a simple method for such a case in the next article.

All the thirteen steps done above are put in a simple form below. [See P. 42 for diminishing the roots.] .

				4·2644
1	1	1	−100	4
	4	20	84	
	5	21	−16000	2
	4	36	11928	
	9	5700	−4072000	6
	4	234	3788376	
	130	5964	−283624000	4
	2	268	256071744	
	132	623200	−27552256000	
	2	8196		
	134	631396		
	2	8232		
	1360	63962800		
	6	55136		
	1366	64017936		
	6	55152		
	1372	6407308800		
	6			
	13780			
	4			
	13784			
	4			
	13788			
	4			
	137920			

For finding the last number in the decimal place we need not continue the process of diminishing. After annexing zeroes, find the integer by trial divisor.

§ 6. Contraction in Horner's Method.

We have seen that if in the last example we were to evaluate the root correct to nine places of decimal, the usual process of annexing zeroes will be too lengthy and beyond the patience of the students. We shall illustrate the above process by taking the last example. We have seen in last example that after having found the root correct to three places of decimal and then multiplying the roots by 10 we get the transformed equation as

$$x^3+137920x^2+6407303800x-27552256000=0.$$

Dividing the L. H. S. by 1000, we get

$$\cdot001x^3+137\cdot92x^2+6407308\cdot8x-27552256=0.$$

We can assume here that the roots of the above equation will not change very much if the fractional parts of the various coefficients are approximated to integers *i. e.* ·8 may be taken equal to 1 and ·92 also equal to 1 and ·001=0. Hence the above equation changes to

$$138x^2+6407309x-27552256=0.$$

Now in order to find the next integer we use the principle of trial divisor and the figure suggested is 4.

The above method can be put as below :—

Instead of annexing zeroes to the coefficients of transformed equation we cut off one figure from the right of the last but one coefficient (trial divisor), two figures from the right of last but two coefficients and three figures from the right of last but three coefficients and neglect the leading coefficient. We shall however mentally take into consideration the effect of figures cut off, *i. e.* fractions greater than ·5 to be approximated to unity less than ·5 to be neglected. After all the figures of the last but two coefficients have been exhausted, the process will reduce to the ordinary process of division and can be continued until all the figures of the trial divisor are exhausted.

Suppose we apply contraction to the last example after we have evaluated the root up to three places of decimals. The transformed equation is

$$x^3+13792x^2+64073088x-27552256=0.$$

Now instead of annexing zeroes we cut off one figure *i. e.* 8 from the right of last but one coefficient, cut off two figures *i. e.* 92 from the last but two coefficients, and neglect the leading coefficient and we are left with

|4·26~~4~~

137~~92~~ 6407308~~8~~ −27552256.

The trial divisor suggests number 4 and we diminish the roots by 4. While multiplying and adding we have to mentally take into consideration the effect of figures cut off. We shall write *a*, *b*, *c*... at places where we have mentally taken the effect of the figures cut off and explain them below.

|4·264, 4, 2, 9, 9, 7, 3, 1

137~~92~~	6407308~~8~~		−27552256		4
	552	(a)	25631444		
	6407861	(b)	−1920812		2
	552		1281688		
	640841~~3~~		−639124		9
	3	(c)	576762	(e)	
1~~37~~	640844	(d)	−62362		9
	3		57676	(f)	
	64084~~7~~		−4686	(g)	7
	6408~~4~~		4486		
~~1~~	640~~8~~		−200		3
	64~~0~~		192		
	6~~4~~		−8		1
			6		
			−2		

Explanation for mentally taking into consideration the effect of figures cut off.

(a) $552=(4\times137)+(4\times\cdot92)=548+3\cdot68=548+4=552.$
$\cdot68$ is approximated to 1.

(b) $6407861=6407308\cdot8+552=6407309+552.$
$\cdot8$ is approximated to 1.

(c) $3=(2\times1)+(2\times\cdot37)=2+\cdot74=3$
$\cdot74$ is approximated to 1.

(d) $640844=640841\cdot3+3=6408444.$
$\cdot3$ is approximated to 0.

Other explanations can similarly be accounted for.

At what stage to begin contraction ?

If we begin to apply contraction at the stage when the trial divisor contains n integers, then we shall be able to find $(n-1)$ figures of the root in addition to those already found out. Suppose we want to find out the root of a given equation correct to ten places of decimal. If after evaluating three places of decimal we want to apply contraction we should first certify that the trial divisor contains 8 digits so that it may give $(8-1)$ *i.e.* 7 additional figures and hence the root will be found correct to ten places of decimal. In case the trial divisor at this stage consists of 7 digits, then by the process of contraction we shall be furnished with $(7-1)$ *i.e.* 6 additional figures and in all we shall find the root correct to 9 places of decimal whereas we want it correct to 10 places of decimal. Hence we should annex zeroes and proceed with one usual step and apply contraction at the next step.

Ex. 2. *Find to seven places of decimal the root of the equation* $x^4-12x+7=0$ *which lies between 2 and 3.*

(Agra 43, 52, 55, 58, Pb. 25, 44)

Let the equation be written in complete form as

$$x^4+0x^3+0x^2-12x+7=0.$$

```
                                                          | 2·04727566
    1  0        0         -12              7             | 0
 A     2        4           8             -8             |
       2        4          -4         -1,0000,0000       | 4
       2        8          24            83891456        |
       4       12 •     20,000,000      -16108544        | 7
       2       12          972864        15493415        |
       6    24,00,00     20972864         -615129        | 2
       2        3216       985792          446264      • |
     800      243216     21958656         -168865        | 7
       4        3232       17479   (a)     156227 (b)    |
     801      246448     2213345          -12638         | 5
       4        3248       17479            11159  (c)   |
 B   808      249696     2230824           -1479         | 6
       4        2496          50            1339 (d)     |
     812          24     2231 32            -140         | 6
       4                      50             134 (e)     |
     816                 223182               -6
                         22318
                         2231
                         223
```

Explanation :—

A. We will first diminish the roots by 2 and then annex zeroes. The trial divisor is greater than the last coefficient and suggests the next number zero. We put zero in the decimal place and multiply the roots again by 10 by anrexing zeros and proceed as usual.

B. At this stage there are 8 digits in the trial divisor and as such we can find seven more decimal places if we apply contraction now. We have already found two decimal places and hence the root will be evaluated correct to nine places of decimal by applying contraction at this stage. We cut off 6 from trial divisor, 96 from last but two coefficients and 1st two coefficient's vanish altogether. We now proceed as if we are dealing with a quadratic.

Note :—Small letters denote that while multiplying and adding we have mentally taken into consideration the effect of digits cut off at the places where these letters are written as in Ex. 1.

Note :—We see that after diminishing the roots by 2 the sign of the last term is changed from + to – showing that we have crossed a root for, the equation has another root which lies between 0 and 1 as $f(0)=+$ive and $f(1)=-$ive.

Ex. 3. *Find the real root of the equation*

$$x^3-x^2-x-2000=0$$

correct to seven decimal places. **(Agra 41, 60 ; Pb. 41)**

We find that $f(10)=-$ive and $f(20)=+$ive *i.e.* a root lies between 10 and 20: We will diminish the roots of the equation by 10 and the transformed equation is $x^3+29x^2+279x-1110=0$. The root of this equation lies between 0 and 10 and since $f(2)=-$ive, $f(3)=+$ive. ; it lies between 2 and 3. We diminish the root by 2 and the roots of transformed equation will lie between 0 and 1 and we shall multiply its roots by 10 by annexing zeroes and proceed as usual.

			10+2·96867240	
1	−1	−1	−2000	10
	10	90	890	
	9	89	−1110	2
	10	190	682	
	19	279		
	10	62	−428000	9
	29	341	395379	
	2	66		
	31	40700		
	2	3231	−32621000	6
	33	43931	28481736	
	2	3312		
	350		−4139264	8
	9	4724300	3818144	
	359	22656	−321120	6
	9	4746956	286542(c)	
	368	22692	−34577	7
	9		33430(d)	
	3770	476964~~8~~	−1147	2
	6	303 (a)	955 (e	
	3776	477268 (b)	−192	4
	6	303	191	
	3782	47757~~1~~	−1	0
	6	4775~~7~~	0	
	378~~8~~	477~~5~~	−1	
	3~~7~~	47~~7~~		
		4~~7~~		
		~~4~~		

Hence the root is 12·96867240.

Explanation

(A) After calculating one decimal place *i. e.* 9 the trial divisor 47243 had only five digits and we could find only (5 − 1) *i. e.* 4 additional places of decimal *i. e.* in all five places. If we were to calculate the root correct to five or four places of decimal we could have applied contraction at this stage, but since we are to find out the root correct to eight decimal places we apply contraction at the next stage B).

(B) After diminishing the roots by 6 we find that trial divisor 4769648 has seven digits and as such it can give us (7 − 1) *i. e.* 6 additional places of decimal and two places of decimal we have already found. Hence if we apply contraction at this stage the root will be known correct upto eight places of decimal.

(a) $303=(8\times37)+(8\times\cdot88)=296+6{\cdot}94=296+7=303.$

(b) $477268=476964{\cdot}8+303=476965+303.$

·8 approximated to 1.

(c) $286543=6\times47757+6\times\cdot1$
$=286542+\cdot6=286543.$

Ex. 4. *$x^3-2x-5=0$ has a root between 2 and 3. Find it correct to 9 places of decimal.* **(Agra 35; '44; Pb. 34)**

			2·094551483	
1	0	−2	−5	2
	2	4	4	
	2	2	−1,000	0
	2	8		
	4	10,00	−1,000,000	9
	2		949329	
A.	60	10,00,00	−50671000	4
		5481	44517584	
	60,0	105481	−6153416	5
	9	5562	5578825	
	609	11104300	−574591	5
	9	25096	(c) 558040	
	618	11129396	−16551	1
	9	25112	(d) 11161	
	6270	1115450~~8~~	−5390	4
	4	314 (a)	4464	
	6274	1115765 (b)	−926	8
	4	314	(e) 893	
B.	6278	111607~~9~~	−33	3
	4		33	
	62~~82~~	11160~~7~~	0	
	~~62~~.	1116~~0~~		
		111~~6~~		
		11~~1~~		

Hence the root correct upto nine places of decimal is 2·094551483.

Explanation.

(A) Since the trial divisor does not suggest any number, we diminish the roots by zero and then multiply the roots by 10 by annexing zeroes.

(B) We apply contraction at this stage as the trial divisor 11154508 contains 8 digits and it will therefore give the root upto 7 more decimal places and three places we have already found. We could not have found the root correct upto nine places of decimal had we applied the contraction at the previous step.

Explanations a, b, c, d, e are as usual.

Example 5. *The equation* $x^4+4x^3-4x^2-11x+4=0$ *has a root between 1 and 2. Find its value correct to eight places of decimals.*

|1·63691357

1	4	−4	−11	4
	1	5	1	−10
	5	1	−10	−60000
	1	6	7	50976
	6	7	−3000	−90240000
	1	7	11496	72690561
	7	1400	8496	−17549439
	1	516	14808	15213090
	80	1916	23304000	−2336349
	6	552	926187	2301606(d)
	86	2468	24230187	−34743
	6	588	935601	25602(e)
	92	305600	2516578$\cancel{8}$(a)	−9141
	6	3129	18936	7680
	98	308729	2535515	−1461
	6	3138	18972	1280
	1040	311867	255448$\cancel{7}$	−181
	3	3147	285 (b)	179 (f)
	1043	31501$\cancel{4}$	255734 (c)	−2
	3	6	285	
	1046	3156	25601$\cancel{9}$	
	3	6	2560$\cancel{1}$	
	1049	3162	256$\cancel{0}$	
	3	6	25$\cancel{6}$	
	10$\cancel{5}\cancel{2}$	31$\cancel{6}\cancel{8}$	2$\cancel{5}$	
		3$\cancel{1}$		

Hence the root correct to eight places of decimal is 1·63691357.

Exercise.

1. *The equation* $x^4+12x+7=0$ *has a root between* -1 *and* 0. *Calculate it correctly to seven places of decimal.* **(Agra 54)**

Since the root of $f(x)\equiv x^4+12x+7=0$ is $-$ive as it lies betwen -1 and 0, hence the root of

$$f(-x)=x^4-12x+7=0$$

will have a root between 0 and 1. **Ans.** $-\cdot5936858$

2. *Find all the real roots of the equation* $x^3-3x+1=0$ *correct to five places of decimal.* **(Agra 53)**

It can be easily seen that $f(x)=0$ has two $+$ive roots, one lying between 0 and 1 and the other between 1 and 2. The $-$ve root of $f(x)=0$ lies between -1, -2.

Ans. 1·53209, 0·34729, $-$1·87938

3. *Find the* $+$*ive root of the equation*

$$2x^3-85x^2-85x-87=0.$$ **(Arga 40)**

Since the last three terms of $f(x)$ are $-$ive, hence the root will have a greater number for its integral part. We find that it lies between 40 and 50 as $f(40)$ and $f(50)$ are of opposite signs. Diminish the roots of $f(x)$ by 40 and then the roots of the transformed equation will lie between 0 and 10 and they are as usual found to lie between 3 and 4 etc. **Ans.** 43·5

4. *Find to three places of decimal the root of the equation* $-3x^2+75x-10000=0$ *which is situated between 9 and 10.*

(Delhi 49, Pb. 40) Ans. 9·886

5. *Find the cube root of 25 correct upto ten decimal places.*

In other words we have to find the root of $x^3-25=0$.

Ans. 2·92401773821

6. *The root of the equation* $x^3-7x+7=0$ *lies between* $3/2$ *and* 2 *; find it correct to four decimal places.* **(Agra 57, Pb. 54)**

Multiply the roots of $f(x)$ by 2 and the root of the transformed equation will lie between 3 and 4 etc.

7. *Find the +ive root of the equation* $x^5-7=0$ *correct to five places of decimal.* **(Pb. 46)** ***Ans.*** 1·47577

8. *Find a root of the equation*

$$x^4+x^3-x^2-10x+3=0$$

which is situated between 0 *and* 1, *correct to four decimal places.*

(Pb. 43) Ans. ·3131

9. *Find the real root of the equation* $x^3+29x-97=0$ *to five places of decimal.* **(Pb. 45) Ans.** 2·68061

10. *Find by Horner's method the cube root of* 30 *correct to seven places of decimal.* **(Pb. 58)**

§ 7. Nearly equal roots.

Suppose $f(x)=0$ has two nearly equal roots say $\alpha\cdot\beta\gamma\delta\ldots.$and $\alpha\ .\beta\gamma\delta'\ldots.$ We also know that in case an equation $f(x)=0$ has two roots equal to α then $f(\alpha)=0$ as well as $f'(\alpha)=0$, when the roots are nearly equal to α then both $f(\alpha)$ and $f'(\alpha)$, will be small quantities. Let us diminish the roots of $f(x)=0$ by α ; we get the transformed equation

$$f(x+\alpha)=0 \quad\text{or}\quad f(\alpha)+xf'(\alpha)+\frac{x^2}{2\,!}f''(\alpha)+\ldots\ldots$$

$$+\frac{x^n}{n\,!}f^n(\alpha)=0 \quad\ldots\qquad\ldots\quad\ldots(1)$$

$$p+qx+rx^2+\ldots\ldots=0$$

where $p=f(\alpha)$, $q=f'(\alpha)$, $r=\dfrac{f''(\alpha)}{2\,!}$ and p, q are quite small.

Suppose that the roots differ from α by a small quantity h.

So that $\alpha+h$ is the actual value of the root, then

$$f(\alpha+h)=0$$

or
$$f(\alpha)+hf'(\alpha)+\frac{h^2}{2\,!}f''(\alpha)+\ldots\ldots=0$$

or
$$p+qh+rh^2+\ldots\ldots=0.$$

Since the equation has two nearly equal roots *i. e.* $f(\alpha)$ and $f'(\alpha)$ or p and q are quite small, we cannot neglect the term rh^2 in comparison to qh as we neglected in the case when the roots were not nearly equal [see page 210]. Hence in this case the transformed equation (1) reduces to a quadratic $p+qx+rx^2=0$ and it has nearly equal roots, say equal to 'k'; then

$$(p+qx+rx^2)\equiv r(x-k)^2\equiv r(x^2-2kx+k^2).$$

Comparing the coefficients, we get

$$\frac{p}{rk^2}=\frac{q}{-2rk}=\frac{1}{1}$$

or
$$k=\frac{-2p}{\ }=\frac{q}{-2r}$$

or
$$k=-\frac{\text{twice the last coefficient}}{\text{last but one coefficient}}$$

$$=-\frac{\text{last but one coefficient}}{\text{twice last but two coefficient}}\cdot$$

In case we find that these two values of k are nearly equal we should feel that the roots have not been separated and they are still equal and we should diminish the root by k and again apply above. In case the values of k given by above are different, we should feel that the roots have been separated and then find the two regions in which the roots lie by the principle that if $f(a)$ and $f(b)$ be of opposite signs a root lies between them. It may also be noted that while calculating the greater root you will cross over the smaller root and as such the sign of the last term will change.

Ex. 1. (a) *The equation* $x^3-7x+7=0$ *has two roots between 1 and 2. Find each of them to four decimal places.* **(Agra 57)**

1. Diminishing the roots of the given equation by 1 we get the transformed equation as $x^3+3x^2-4x+1=0$ whose roots lie between 0 and 1.

2. Multiplying the roots by 10 we get the transformed equation $x^3+30x^2-400x+1000=0$ whose roots lie between 0 and 10,

In order to find whether the roots have been separated or not,

$$k=\frac{-2\ \text{last coeff.}}{\text{last but one}}=\frac{-2000}{-400}=5$$

or
$$=\frac{-\text{last but one}}{2.\text{last but two}}=\frac{400}{60}=7\ \text{nearly.}$$

Since the values of k are different we should feel that the roots have been separated. By calculating $f(a)$ and $f(b)$ the two roots are found to lie between 3, 4 and 6, 7. It will be seen that while calculating the greater root the sign of the last term will change whereas while calculating the smaller root the sign will remain unchanged as shown below :—

Smaller root

```
                                          |1·3568
1   30          -400          1000
     3            99          -903
    ----        ------
    33          -301           97000
     3           108          -86625
    ----        ------        ---------
    36          -19300         10375000
     3            1975        -9048984
    ----        ------        ---------
   390          -17325         126016000
     5            2000
    ----        --------
   395          -1532500
     5             24336
    ----        --------
   400          -1508164
     5             24372
    ----        ----------
  4050          -147379200
     6
    ----
  4056
     6
    ----
  4062
     6
    -----
  40680
```

Having calculated upto three places the fourth place of decimal is found to be 8 by the principle of trial divisor.

Greater root

```
                                          |1·6920
1   30          -400          1000
     6           216         -1104
    ----        ------       ---------
    36          -184         -104000*
     6           252          100809
    ----        ------       ---------
    42           6800        -3191000
     6           4401         3156888
    ----        ------       ---------
   480           11201       -34112000
     9           4482
    ----        ---------
   489           1568300
     9             10144
    ----        ---------
   498           1578444
     9             10148
    ----        ---------
  5070           158859200
     2
    ----
  5072
     2
    ----
  5074
     2
    -----
  50760
```

---

* The sign of the last term is changed. Having calculated upto three places the fourth place of decimal is found to be zero by the principle of trial divisor.

(b) *The equation $x^4-14x^2+16x+9=0$ has two roots between 2 and 3. Find them to three decimal places.* **(Pb. 37)**

1. Diminishing the roots of the equation by 2, we get the transformed equation as $x^4+8x^3+10x^2-8x+1=0$ whose roots lie between 0 and 1.

2. Multiplying the roots of above by 10 we get the transformed equation $x^4+80x^3+1000x^2-8000x+10000=0$. We do not know whether the roots are still equal in decimal places *i.e.* they have been separated or not.

It k be nearly equal to next digit of decimal place then

$$k=\frac{-2\text{ last coeff.}}{\text{last but one}}=\frac{-20000}{-8000}=2\text{ nearly}$$

or
$$=\frac{-\text{last but one coeff.}}{2.\text{last but two}}=\frac{8000}{2000}=4.$$

Since the values of k given, are different, we should feel that the roots have been separated and their decimal places are not equal.

Now $x^4+80x^3+1000x^2-8000x+10000=0$.

It has two roots in different regions. By the principle of trial divisor the next figure by which the roots be diminished is 1 *i. e.* the root lies between 1 and 2. The other root is obtained to be between 4 and 5 as $f(4)$ and $f(5)$ are of opposite signs and while calculating this root the sign of the last term will change as you will be crossing over the smaller root. Proceeding as usual in Horner's method for the different roots we obtain them to be **2·161 and 2·414.**

Ex. 3. *Find to seven places of decimal the value of each of the two roots of the equation*

$$3x^4-61x^3+127x^2+220x-520=0$$

which lie between 2 and 3.

1. Diminishing the roots by 2 and then multiplying the roots by 10, the transformed equation is

$$3x^4-370x^3-16700x^2+92000x-120000=0 \quad \ldots\ldots(1)$$

$$k=+\frac{2\,(120000)}{92000}=2{\cdot}6 \quad \text{or} \quad \frac{92000}{2\,(16700)}=2{\cdot}6.$$

As the values of k are nearly equal, the roots have not been separated and we diminish the roots by 2 as suggested by the value of k.

2. Diminishing by 2 and multiplying by 10, we get the transformed equation as

$$3x^4-3400x^3-1884800x^2+20856000x-57120000=0 \quad \ldots(2)$$

$$K=\frac{2\,(57120000)}{20856000}=5 \quad \text{or} \quad \frac{20856000}{2\,(1884800)}=5.$$

As the values of k are nearly equal we feel that the roots have not been separated and we diminish the roots by 5.

3. Diminishing by 5 and multiplying by 10, we get the transformed equation as

$$3x^4-3400x^3-1936250x^2+1750000x-390625=0 \quad \ldots.(3)$$

In order to find k we need not multiply the roots by 10 as we shall apply contraction.

$$k=\frac{2\,(390625)}{1750000}={\cdot}44=\frac{1750000}{2\,(1936250)}={\cdot}45.$$

The above shows that the roots have still not been separated and we diminish the roots by 4 by applying contraction and the resulting equation is

$$3x^3+19404x^2-19936x+641=0 \ldots \ \ldots \ \ldots(4)$$

Here $k=\dfrac{2\,(641)}{19936}={\cdot}06$ or $\dfrac{19936}{2\,(19404)}={\cdot}5.$

As the values of *k* are different, we conclude that the roots have been separated. By trial it is seen that the equation (4) has one root between 0 and 1 whereas the other root lies between 9 and 10 and we proceed to find as usual shown below.

Smaller Root

$$\begin{array}{llll} & & & |\underline{2{\cdot}2540333} \\ \cancel{3} & \dot{1}940\!\!\cancel{0}\!\cancel{4} & -1993\cancel{6} & 641 \\ & 1\cancel{9}\cancel{4} & -199\cancel{3} & 641 \\ & & 6\ \text{(a)} & -579 \\ & & -193 & 62 \\ & & 6 & -56\ \text{(b)} \\ & & -18\cancel{7} & 6 \\ & \cancel{1} & & 5\ \text{(c)} \\ & & 1\cancel{8} & 1 \\ & & \cancel{1} & \end{array}$$

It should be noted that the last figure is 3. Though the numbers 1 and 6 suggest 6 but 1 is actually 1·8 and when multiplied by 6 it would be 10·8=11 nearly but when multiplied by 3 it becomes 5·4=5 nearly.

Greater Root

$$\begin{array}{llll} & & & |\underline{2{\cdot}2549942} \\ \cancel{3} & 194\cancel{0}\cancel{4} & -1993\cancel{6} & 641 \\ & & 1746 & -2223 \\ & & -247 & -1582 \\ & & 1746 & 1503 \\ & & 149\cancel{9} & -79 \\ & 1\cancel{9}\cancel{4} & 17\ \text{(a)} & 74\ \text{(c)} \\ & & 167\ \text{(b)} & -5 \\ & & 17 & 4 \\ & & 18\cancel{4} & -1 \\ & \cancel{1} & & \\ & & 1\cancel{8} & \\ & & \cancel{1} & \end{array}$$

As in smaller root, the last figure is 2.

Hence the roots are

2·2540333 and 2·2549942.

Ex. 4. *The equation* $x^4+8x^3-70x^2-144x+936=0$ *has two roots which lie between 4 and 5. Find them correct to three places of decimal.*

4·242, 4·246.

§ 8. Lagrange's method of expressing the root of a numerical equation in the form of a continued fraction :— (Agra 45)

Let the equation be $f(x)=0$ and suppose its root lies between two consecutive integers p and $p+1$. Let the root be taken to be $p+\frac{1}{y}$. $\therefore\ f\left(p+\frac{1}{y}\right)=0$.

In order to get $f\left(p+\frac{1}{y}\right)$, we first diminish the roots of the equation by p and then from the transformed equation find other equation whose roots are reciprocal of the roots of the equation. This equation will give us the value of y. By trial we find that $\phi(y)=0$ has a root between q and $q+1$; let it be $q+\frac{1}{z}$. Diminish the roots by q and then from the transformed equation find another equation whose roots are reciprocals of the roots of the transformed equation. This equation $\psi(z)=0$ has a root between r and $r+1$ etc.

Hence the root of the given equation is

$$p+\cfrac{1}{q+\cfrac{1}{r+\dots}}$$

Ex. *Find in the form of a continued fraction the +ive root of the equation* $f(x)=x^3-6x-13=0$.

Clearly $f(3)=-$ive and $f(4)$, +ive and as such the root lies between 3 and 4. Let it be $3+\frac{1}{y}$. Diminishing the roots by 3, we get the equation $x^3+9x^2+21x-4=0$. Its root is $\frac{1}{y}$.

Hence y is a root of $\phi(y) \equiv 4y^3 - 21y^2 - 9y - 1 = 0$.

The +ive root of $\phi(y) = 0$ lies between 5 and 6.

Since $\phi(5) = -$ive and $\phi(6) = +$ive.

Let it be $5 + \frac{1}{z}$. Diminishing the root by 5, we get the equation as $4y^3 + 39y^2 + 81y - 71 = 0$ whose root is $1/z$. Hence z is a root of the equation

$$71z^3 - 81z^2 - 39z - 4 = 0.$$

Its root lies between 1 and 2 and let it be $1 + \frac{1}{u}$ etc.

DELHI UNIVERSITY B. A. HON'S PAPER SOLVED

1948

1. (i) A root α of the equation $3x^3-10x^2+7x+10=0$ is connected with a root α' of the equation $x^3-x^2-17x+65=0$ by the relation $a\alpha'+\alpha'-\alpha+1=0$. Using this fact, solve the two given equations completely.

(ii) If $\alpha_1, \alpha_2 \dots, \alpha_n$ are the roots of the equation $x^n+nax-b=0$, show that

$$(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\dots(\alpha_1-\alpha_n)=n(\alpha_1^{n-1}+\alpha).$$

Q. 13 P. 16.

2. Show that if p and q $(p\neq q)$ be the roots of the equation $(a_0z+a_1)(a_2z+a_3)=(a_1z+a_2)^2$, then the cubic equation $a_0z^3+3a_1z^2+3a_2z+a_3=0$ can be reduced to the form

$$A(z-p)^3+B(z-q)^3=0.$$

Hence or otherwise solve the equation

$$46z^3+72z^2+18z-11=0.$$

§ 3 P. 22. Proceed as in Ex. 1 of 2 P. 125–126.

3. Show how to solve a biquadratic equation by resolving it into two quadratic factors.

Solve the equation $x^4+12x+3=0$.

§ 7 P. 140, just as Ex. 9 P. 143.

4. (i) Find the sum of the fifth powers of the roots of the equation $x^4-7x^2+4x-3=0$.

(ii) Calculate $\Sigma\alpha_1^2\alpha_2^2$ for the general equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0.$$

Simple, Ex. 1 P. 110.

5. State and prove Sturm's Theorem on the separation of the roots of an equation.

Find the number and situation of the real roots of the equation $x^3-3x+1=0$.

§ 3 P. 179, Ex. 2 P. 186.

6. (i) Apply Newton's Method of Divisors to find the integral roots of $3x^4-23x^3+35x^2+31x-30=0$.

(ii) Find the positive root of the equation $x^3+x^2+x-100=0$ correct to four decimal places.

Ex. 2 P. 207, Proceed as in Ex. 2, 3 P. 220, 222.

1949

1. (i) The sum of two roots of the equation $x^4-8x^3+19x^2+4\lambda x+2=0$ is equal to the sum of the other two. Find λ and solve the equation.

(ii) Find the multiple roots of the equation

$$x^4+12x^3+32x^2-24x+4=0.$$

Hint : Find the H. C. F. of $f(x)$ and $f'(x)$ etc.

Q. 9. (b) P. 29.

2. Obtain the roots of the cubic $ax^3+3bx^2+3cx+d=0$.

If the cubic $x^3+3Hx+G=0$ has roots α, β, γ, show that the cubic $x^3+9Hx^2-27(G^2+4H^3)=0$ has roots

$$(\alpha-\beta)(\alpha-\gamma),\ (\beta-\gamma)(\beta-\alpha),\ (\gamma-\alpha)(\gamma-\beta).$$

§ 1 P. 112 or § 3 P. 122, Ex. 8 (b) P. 74.

3. (i) Prove that the expression

$$u \equiv z^4+6Hz^2+4Gz+a^2I-3H^2$$

can be written as the product of the two factors

$$z^2+3H+2k^2 \pm \left(2kz-\frac{G}{k}\right)$$

provided that k^2+H is a root of the equation

$$4\phi^3-a^2I\phi+a^3J=0,$$

where $\qquad G^2+4H^3=a^2\,(HI-aJ).$

(ii) Show that the roots of the biquadratic

$$ax^4+4bx^3+4dx+e=0$$

have only two distinct values if

$$\frac{ad^2}{b^2e}=\frac{3bd}{bd-ae}=\pm 1,$$

and distinguish between the two cases.

Ex. 9 P. 149.

4. (i) If α, β, γ are the roots of the equation

$$x^3+px+q=0,$$

prove that $\dfrac{\alpha^5+\beta^5+\gamma^5}{5}=\dfrac{\alpha^3+\beta^3+\gamma^3}{3}\times\dfrac{\alpha^2+\beta^2+\gamma^2}{2}$.

(ii) Apply Sturm's Theorem to the analysis of the equation $x^4-4x^3+7x^2-6x-4=0$.

Ex. 5 (b) P. 103, Ex. 8 (d) P. 192.

5. (i) Find by *Horner's Method* the positive root of the equation $16x^3-20x^2-50x-375=0$.

(ii) Solve $2x^3-31x^2+112x+64=0$ given that all the roots are *commensurable.*

Q. 3. P. 208.

1950

1. Calculate Sturm's remainders for the biquadratic

$$z^4+6Hz^2+4Gz+a^2I-3H^2=0$$

and discuss the nature of the roots of that equation with the help of those remainders.

Ex. 13 P. 193, § 5 P. 196.

2. Solve completely the equation

$$2x^4+x^3-2x^2-4x-3=0.$$

given that two roots of its are commensurable.

Find to four decimal places the negative root of the equation

$$x^3-x^2+12x+24=0.$$

Hint. (i) Multiply the roots by 2 and apply Newton's method of divisors. (ii) −ive root of $f(x)=0$ is +ive root of $f(-x)=0$.

3. If

$$a(ax^4+4bx^3+6cx^2+4dx+e)\equiv(ax^2+2px+r)(ax^2+2qx+s)$$

and $r+s=2(c-2\phi)$, find the cubic equation giving ϕ.

Solve $2x^4+6x^3-3x^2+1=0.$

Ex. 9 P. 149, Ex. 4 P. 148.

✓ 4. Given that

$$x+y+z=3,$$
$$x^2+y^2+z^2=5,$$
$$x^3+y^3+z^3=7,$$

find the value of $x^4+y^4+z^4$.

If α, β, γ and δ be the roots of the equation $x^4-x+1=3$, form the equation whose roots are

$$\alpha(1+\alpha^2),\ \beta(1+\beta^2),\ \gamma(1+\gamma^2) \text{ and } \delta(1+\delta^2).$$

Ex. 8 (b) P. 104.

✓ 5. Find the condition that the roots of the equation

$$x^3+3px^2+3qx+r=0$$

may be in (i) A. P., (ii) G. P., (iii) H. P. Solve the equation in case (ii).

If $x+a_r y+a_r^2 z+a_r^3 t=a_r^4$ $(r=1, 2, 3, 4)$, find x, y, z and t.

Q. 6 P. 27, Proceed as in Q. 14 P. 33.

1951

1. (a) If all the roots of the equation

$$x^n+p_1x^{n-1}+\ldots+p_n=0$$

are real and negative, show that

$$\left(\frac{p_1}{n}\right)^n \geqslant p_n.$$

(b) The coefficients in the equation

$$a_0 y^n+a_1 y^{n-1}+\ldots+a_n=0$$

are connected by the relation

$$\frac{a_0}{n+1}+\frac{a_1}{n}+\ldots+\frac{a_n}{1}=0.$$

Show that it has at least one root between 0 and 1.

2. Prove Sturm's theorem on the separation of the real roots of an equation whose roots are unequal.

Prove that the equation

$$x^5-x-16=0$$

has two pairs of complex roots.

§ 3 P. 179, Ex. 10 P. 193.

3. (a) Find the value of the symmetric function

$$\Sigma\frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2}$$

of the roots of the equation

$$x^n+p_1x^{n-1}+\ldots+p_n=0.$$

(b) The distances of three points A, B, C on a straight line from a fixed origin O on the line are the roots of the equation

$$ax^3+3bx^2+3cx+d=0\ ;$$

find the condition that one of the points A, B, C should bisect the distance between the other two.

Ex. 9 P. 95, Q. 7 P. 27.

4. Give any method of solving the biquadratic equation

$$ax^4+4bx^3+6cx^2+4dx+e=0,$$

and apply it to find the roots of

$$x^4-5x^2-6x-5=0.$$

§ 4 P. 131, § 7 P. 140, § 8 P. 144.

5. (a) If $f(x)$ be a rational integral algebraic function of x and $\alpha, \beta, \gamma \ldots .$ be the roots of $f(x)=0$, prove that

$$\frac{f'(x)}{f(x)}=\frac{1}{x-\alpha}+\frac{1}{x-\beta}+\frac{1}{x-\gamma}+\ldots$$

Hence find the sum of the fourth powers of the roots of the equation

$$x^5+px^4+qx^2+i=0.$$

(b) If $f(x)$ be a rational algebraic function of x and the roots of $f(x)=0$ be all real, prove that the roots of

$$f(x).f'(x)-\{f'(x)\}^2=0$$

are all imaginary.

See P. 99 result 2. Simple.

1952

1. (a) Find the relation connecting p, q, r in order that the cubic

$$x^3-px^2+qx-r=0$$

should have its roots in harmonic progression.

(b) If the roots (α, β, γ, δ) of the equation

$$ax^4+4bx^3+6cx^2+4dx+e^2=0$$

be so related that $\alpha-\delta$, $\beta-\delta$, $\gamma-\delta$ are in harmonic progression, prove that $ace+2bcd-ad^2-b^2e-c^2=0$.

Q. 5 (a) P. 26, Ex. 2 P. 140.

2. (a) Explain any method of solving the cubic equation $x^3+px+q=0$.

(b) Show in any way you please that

$$x^3+x^2-2x-1=0$$

has three real roots, and determine the value of the greatest root to three decimal places.

§ 1 P. 112, use Sturm's Theorem.

3. State and prove a method for obtaining the sum of the nth powers of the roots of a given equation.

If α, β, γ... are roots of an equation, and if S denotes $\Sigma\alpha^r$, prove that

$$\Sigma\alpha^3\beta^2\gamma=S_1S_2S_3+S_1S_5-S_2S_4+2S_6-S_3^2.$$

§ 2 P. 99, P. 111.

4. (a) If all the coefficients in the equation

$$x^n+p_1x^{n-1}+\ldots.+p_n=0$$

be whole numbers, prove that the equation cannot have a fractional root.

(b) If the roots of the equation $x^n-1=0$ are 1, α, β, γ...., show that

$$(1-\alpha)(1-\beta)(1-\gamma)\ldots.=n.$$

§ 3. 198, Q. 9 P. 15.

5. (a) If the equation whose roots are the squares of the roots of the cubic

$$x^3-ax^2+bx-1=0$$

is identical with the cubic, prove that either $a=b=0$, or $a=b=3$ or a, b are the roots of

$$y^2+y+2=0.$$

(b) Find the number and situation of the real roots of

$$x^6-2x^2+3x-4=0.$$

§ 2 Page 198, Ex. 1 page 185.

1953

1. (a) Increase by 7 the roots of the equation

$$3x^4+7x^3-15x^2+x-2=0,$$

and get the resulting equation.

(b) Solve the equation

$$x^3+3x^2+4x-10=0$$

by removing the second term.

Just as in Ex. 2 P. 44, remove the second term and proceed as in Ex. 1 and 2 P. 116, 117.

2. Compute H and G for the cubic

$$x^3+6x^2+12x-19=0$$

and H, G, I, J for the quartic

$$2x^4+16x^3-2x^2+x-12=0,$$

where H, G, I and J have their usual meanings.

Discuss also the nature of the roots of the above quartic.

§ 8 P. 51, use Sturm's theorem or § 5 P. 196.

3. Define the order and weight of a symmetric function, and explain their use in calculating the value of a symmetric function of the roots of an algebraic equation in terms of the coefficients,

Calculate the value of $\Sigma\alpha_1^2\alpha_2^2\alpha_3^2$, where $\alpha_1, \alpha_2 \ldots \alpha_5$ are the fifth roots of -1.

§ 3 P. 109, See page 110 and put p=2 in $\Sigma\alpha^p\beta^p\gamma^p$

4. State and prove Sturm's theorem for the case of unequal roots.

Determine the integer nearest to each of the roots of

$$x^5-5x^4+9x^3-9x^2+5x-1=0.$$

§ 3 P. 179. See solved examples of Sturm's theorem.

5. Prove that if α be any root of

$$x^3-6x^2+11x-6=0,$$

so is $\beta=(7\alpha-13)/(3\alpha-5)$; and that a homographic relation of similar properties exists in the case of every cubic equation.

1954

1. (a) Find the value of the symmetric function

$$\Sigma\frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2}$$

of the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots p_{n-1}x+p_n=0.$$

(b) The distances of three points A, B, C on a straight line from a fixed origin O on the line are the roots of the equation

$$ax^3+3bx^2+3cx+d=0\ ;$$

find the condition that one of the points A, B, C should bisect the distance between the other two.

Ex. 9 (i) P. 95, Q. 7 P. 27.

2. State and prove Sturm's theorem for locating the real roots of an algebraic equation, no two of the roots being equal.

Locate the real roots of $x^4-8x^3+25x^2-36x+8=0.$

§ 3 P. 179, Q. 8 (c) P. 192.

3. Obtain Newton's formulae to determine the sum of the kth powers of the roots of an equation of the nth degree.

Hence express $\Sigma\alpha^2\beta^2\gamma$ in terms of the coefficients of the general equation of the nth degree, where $\alpha, \beta, \gamma \ldots$ are its roots.

§ 2 page 99, Ex. 1 (iv) page 110.

4. Discuss briefly, indicating the necessary conditions, the cases in which a biquadratic equation

$$x^4+px^3+qx^2+rx+s=0$$

is soluble by means of square roots only.

✓Solve in any way

$$x^4+2x^3+3x^2+4x+4=0.$$

§ 5 page 133, proceed as in Ex. 3 page 147.

5. Explain Newton's method of approximating to the numerical value of any real root of an algebraic equation.

The equation $x^4-3x^2+75x-10000=0$ has a root between 9 and 10. Prove this and find its value to four significant figures.

See page 212, Q. 4, P. 230.

PUNJAB UNIVERSITY B. A. PAPERS SOLVED

1951

1. (a) Establish the relations between the roots and coefficients of a general equation of the *n*th degree.

(b) Find the condition that the roots of the cubic

$$x^3-px^2+qx-r=0$$

may be in harmonical progression.

§ 1 P. 19, Q. 4 (a) P. 25.

2. (a) Explain Descarte's method of solving the biquadratic equation $x^4+px^2+rx+s=0$.

(b) Solve the equation

$$x^4+12x-5=0.$$

§ 7 P. 140, $-1\pm\sqrt{2}$, $1\pm 4i$.

3. (a) Prove that for an equation, all of whose coefficients are integers, the coefficient of the highest power of the variable being unity, an integral root is a divisor of the constant term.

Hence develop the proof of Newton's method of divisors for finding the integral roots of a given equation with integral coefficients.

(b) Find the commensurable roots of

$$x^5-29x^4-38x^3+31x^2-32x+60=0.$$

§ 2 P. 198, § 3 P. 200, Ex. 4 P. 207.

4. If α, β, γ be the roots of the cubic

$$ax^3+3bx^2+3cx+d=0,$$

prove that the equation in y whose roots are

$$\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha};\ \frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta};\ \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}$$

is obtained by the transformation $axy+b\,(x+y)+c=0$.

Hence form the equation with the above roots.

Ex. 5 P. 71, Ex. 6 P. 72.

1952

1. (a) Correct the mistakes (if any) in the following statements : For a correct statement write 'C' only.

(i) An equation $f(x)=0$ in which the coefficient of the first term is unity and the coefficients of the other terms rational numbers, cannot have a commensurable root which is not a whole number.

(ii) Incommensurable roots can be of the following three types only :—

1. Roots involving $\sqrt{(-1)}$.
2. Roots expressible as interminable decimal fraction.
3. Roots expressible as recurring decimal fractions.

(iii) If n be the degree of $f(x)$ and μ and μ' the numbers of changes of sign in $f(x)$ and $f(-x)$ respectively, then if $\mu+\mu' < n$, the equation $f(x)=0$ has exactly $n-(\mu+\mu')$ imaginary roots.

(iv) If two numbers a and b substituted for x in polynomial $f(x)$ give results with the same sign, no real root lies between them and if they give of opposite signs then only one real root lies between them.

(v) There can be polynomials having the following factors only :—

1. $(x-3)(x-5)(x-7)^2$.

2. $[x-3\sqrt{(-5)}][x-\sqrt{(-5)}+3]$.

3. $(x+7-5i)^2(x+7+5i)$.

(b) Find, in terms of the coefficients of the biquadratic equation $a_0x^4+4a_1x^3+6a_2x+4a_3x+a_4=0$, the value of the following symmetric function

$$(\beta-\gamma)^2(\alpha-\delta)^2+(\gamma-\alpha)^2(\beta-\delta)^2+(\alpha-\beta)^2(\gamma-\delta)^2,$$

where α, β, γ δ are the roots.

(i) In place of rational we should have **integral coefficients.**

§ 2 P. 198.

(ii) (1) roots which are not imaginary.

(2 and 3) **See § P. 198 .**

(iii) Use the word **at least** in place of **exactly.**

See P. 152 and 153.

(iv) **§ 2 page 152.** 'No real root or an even number of real roots lie between a and b.'

'At least one or an odd number of roots lies between them.'

(v) 1. Yes.

2. Imaginary roots occur in conjugate pairs

$$[x-3+\sqrt{(-5)}][x-3-\sqrt{(-5)}].$$

3. $(x+7-5i)^2(x+7+5i)^2$. An equation cannot have three imaginary roots.

(B) See Ex. 1 (i) page 150.

✓ (a) Explain Cardan's method of solving the cubic equation

$$x^3+qx+r=0.$$

(b) Solve the equation

$$x^3-15x^2-33x+847=0.$$

§ 1 P. 112, Ex. P. 113.

3. Find to four decimal places the root of the equation

$$x^4+4x^3-4x^2-11x+4=0$$

which lies between 1 and 2.

Q. 5 P. 226.

4. Find the condition that the roots $\alpha, \beta, \gamma, \delta$ of

$$x^4+px^3+qx^2+rx+s=0$$

should be connected by the relation $\alpha\beta=\gamma\delta$.

Hence or otherwise find the cubic whose roots are

$$\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta},\ \frac{\gamma\alpha-\beta\delta}{\gamma+\alpha-\beta-\delta},\ \frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta}.$$

Q. 10 (a) P. 29, Ex. 7 P. 74.

1953

1. (a) Explain Cardon's method of solving the cubic

$$x^3+qx+r=0.$$

(b) Prove that roots of

$$x^3-3x+1=0$$

are $$2\cos\frac{2\pi}{9},\ 2\cos\frac{8\pi}{9},\ 2\cos\frac{14\pi}{9}.$$

§ 1 P. 112, Ex. 2 P. 117.

2. (a) The roots of the cubic

$$x^3+qx+r=0$$

are α, β, γ. Form the equation woose roots are

$$\beta^2+\beta\gamma+\gamma^2,\ \gamma^2+\gamma\alpha+\alpha^2,\ \alpha^2+\alpha\beta+\beta^2.$$

(b) Find the value of

$$\frac{\beta^2+\gamma^2}{\beta\gamma}+\frac{\gamma^2+\alpha^2}{\gamma\alpha}+\frac{\alpha^2+\beta^2}{\alpha\beta},$$

where α, β, γ are the roots of the cubic

$$x^3+px^2+qx+r=0.$$

Ex. 2 (iii) P. 68, Ex. 3 (i) P. 85.

3. (a) Show that all the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0$$

can be obtained when they are in arithmetical progression.

(b) Solve the equation

$$x^4+15x^3+70x^2+120x+64=0$$

whose roots are in geometrical progression.

Q. 15 P. 34, Q. 10 (a) P. 29.

4. Find correctly to five decimal places the cube-root of 25 by Horner's process.

Q. 6 P. 226.

1954

1. (a) Prove that in an equation with real coefficients imaginary roots occur in pairs.

(b) Show that the equation

$$\frac{A^2}{x-a}+\frac{B^2}{x-b}+\frac{C^2}{x-c}+\ldots+\frac{H^2}{x-h}=k$$

has all real roots.

§ 2·2 P. 4, Q. 6 (b) P. 14.

2. (a) Find the condition that the roots of

$$x^3+px^2+qx+r=0$$

may be in harmonical progression.

(b) Solve $6x^3-11x^2-3x+2=0$, given that its roots are in harmonical progression.

See Q. 5 P. 25.

3. (a) Explain Descarte's method of solving the general biquadratic equation.

(b) Solve the equation

$$x^4-2x^2+8x-3=0.$$

§ 7 P. 140, Ex. 7 P. 86.

4. Calculate by Horner's method the root lying between $\frac{3}{2}$ and 2, of the equation $x^3-7x+7=0$, correct to four places of decimal.

Ex. 7. P. 226.

1955

1. (a) Explain a method for finding the supperior and inferior limits of the positive and negative roots of the equations $f(x)=0$. Apyly your method to find a superior limit of the roots of the equations $x^4-x^3-2x^2-4x-24=0$.

(b) Find all the commensurable roots of the equation

$$x^5-29x^4-31x^3+31x^2-32x+60=0.$$

§ 6, 7 P. 175–176, Ex. 4 P. 207.

2. (a) If α, β, γ are the roots of the equation

$$x^3-px^2+qx-r=0,$$

find the value of (i) $\Sigma\alpha^2\beta^2$, (ii) $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$.

(b) If α, β, γ, δ, ϵ are the roots of the equation

$$x^5+ax^3+bx^2+c=0,$$

prove that $$\Sigma\alpha^3\beta^3\gamma=7ac-a^2b.$$

Ex. 1 (v) P. 84, see P. 87.

3. (a) Diminish the roots of $a_0x^3+3a_1x^2+3a_2x+a_3=0$ by h and find the condition in order that the second and third term be removed simultaneously.

(b) If α, β, γ are the roots of the cubic $x^3+3x+2=0$, find the equation whose roots are

$$(\alpha-\beta)(\alpha-\gamma),\ (\beta-\gamma)(\beta-\alpha),\ (\gamma-\alpha)(\gamma-\beta).$$

Hence show that the cubic $x^3+3x+2=0$ has two imaginary roots. Verify this by using Descarte's rule of signs.

§ 9 P. 52, H=0, Ex. 8 (a) P. 74.

4. (a) Explain Descarte's method of solving the biquadratic equation $x^4+qx^2+rx+s=0$.

(b) Solve the equations $x^4-12x-5=0$.

§ 7 P. 140, proceed as in Ex. 1 P. 141.

$$p^2=4 \text{ etc. } -1\pm 2i,\ 1\pm\sqrt{2}.$$

1956

1. Find the equation whose roots are the squares of the roots of the quartic equation

$$x^4+bx^2+cx+d=0.$$

Hence or otherwise show that if $\alpha, \beta, \gamma, \delta$ are four numbers such that

$$\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}+\sqrt{\delta}=0,$$

then
$$64\alpha\beta\gamma\delta-[4\Sigma\alpha\beta-(\Sigma\alpha)^2]^2=0.$$

See Q. 1 (b) P. 59, See Ex. 1 (d) P. 60.

2. (a) Find the condition that two roots of the cubic equation

$$x^3-bx^2+cx-d=0$$

be equal.

(b) If x, y, z are three numbers such that

$$x+y+z=1,$$

$$x^2+y^2+z^2=2,$$

and
$$x^3+y^3+z^3=3,$$

form the equation whose roots are x, y, z and hence or otherwise find the value of

$$x^5+y^5+z^5.$$

Case III. P. 81, or Ex. 4 alternative method P. 128, see Q. 8 (c), P. 104.

1. (a) Explain Cardan's method of solving the cubic equation

$$x^3+qx+r=0.$$

(b) Solve the equation $x^3-21x-344=0$.

§ 1 P. 112, see Ex. 1 (i) P. 117.

4. Find correct to four decimal places the real root lying between 0 and -1 of the equation $x^3+x+1=0$.

Proceed as in solved Ex. 4 P. 224 etc.

AGRA UNIVERSITY M. A. AND M. Sc. PAPERS SOLVED

1951

1. (a) Find the equation whcse roots are the cubes of the roots of

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0.$$

(b) Solve the equation

$$x^4-10x^3+27x^2-10x-16=0,$$

it being given that the sum of two of its roots is 5.

Hint. Sum of two roots = sum of the other two = 5.

Ex. 2 (a) P. 60, Q. 9 P. 27.

2. Determine the method of finding the sum of the rth powers of the roots of an algebraic equation.

If $a, b, c, \ldots$ are the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots=p_{n-1}x+p_n=0,$$

show that

$$\frac{a^2}{b}+\frac{b^2}{a}+\frac{a^2}{c}+\frac{c^2}{a}+\frac{b^2}{c}+\frac{c^2}{b}+\ldots=p_1-\frac{p_{n-1}(p_1{}^2-2p_2)}{p_n}.$$

Hint. $\Sigma\,\frac{\alpha^2}{\beta}=\Sigma\alpha^2.\Sigma\,\frac{1}{\alpha}-\Sigma\alpha$ etc.

§ 2 P. 99, See Ex. 9 (iv) P. 95.

3. (a) Give Descarte's method of resolving the quartic

$$ax^4+4bx^3+6cx^2+4dx+e$$

into quadratic factors.

(b) If $\alpha, \beta, \gamma, \delta$ be the roots of the above quartic equated to zero, show that

$$f'(\alpha)+f'(\beta)+f'(\gamma)+f'(\delta)=-\frac{32G}{a^2},$$

where G has its usual meaning.

(a) § 7 P. 140, (b) Ex. 15 P. 169, Q. 3 P. 185.

4. (a) Give Newton's method for finding an upper limit to the roots of a given equation, and use it to find the integral part of the greatest root of

$$x^3-10x^2-11x-100=0.$$

(b) Locate the position of the real root of the equation

$$x^3+29x-97=0$$

and explain fully how Horner's method can be used to determine a real root of an equation correct to any number of decimal places.

P. 177, § 5 P. 209 and Ex. 10 P. 228.

1952

1. (a) Prove that if two numbers a and b, when substituted for x in the polynomial $f(x)$, give results with contrary signs, an odd number of real roots of the equation $f(x)=0$ lie between them; and if they give results with the same sign, either no real root or an even number of real roots lie between them.

(b) (i) Find the nature of the roots of the equation

$$x^4+15x^2+7x-11=0.$$

(ii) If α, β, γ be the roots of the cubic

$$x^3+3qx+r=0,$$

form an equation whose roots shall be

$$(\alpha-\beta)^2,\quad (\beta-\gamma)^2,\quad (\gamma-\alpha)^2.$$

§ 2 P. 152, Ex. 1 P. 155, § 13 P. 78.

2. Explain Ferrari's method of resolving the quartic

$$ax^4+4bx^3+6cx^2+4dx+e$$

into its quadratic factors.

Hence solve the equation

$$x^4+2x^3-7x^2-8x+12=0.$$

§ 8 P. 144, Ex. 3 P. 147.

3. (a) Prove that every rational symmetric function of the roots of an algebraic equation can be expressed rationally in terms of the coefficients.

(b) If α, β, γ be the roots of the cubic

$$x^3+px^2+qx+r=0,$$

calculate $\Sigma\alpha^3\beta^3$.

§ 3 P. 109, Ex. 4 (a) P. 102.

4. (a) State Sturm's theorem on the location of real and unequal roots of an equation.

(b) Use Sturm's method to show that the equation

$$x^4-12x+7=0$$

has a root between 2 and 3. Find it by Horner's method to 4 places of decimals.

§ 3 P. 179, Ex. 7 P. 192, Ex. 2 P. 220.

1953

1. (a) If an equation $f(x)=0$, whose coefficients are all real quantities, have for a root the imaginary expression $\alpha+\beta\sqrt{(-1)}$, prove that it must also have for a root the conjugate imaginary expression $\alpha-\beta\sqrt{(-1)}$.

(b) If α, β, γ be the roots of the cubic

$$ax^3+3bx^2+3cx+d=0,$$

form an equation whose roots are

$$\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha},\ \frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta},\ \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}.$$

§ 2·3 P. 4, Ex. 6 (a) P. 72.

2. Explain Cardan's method of solving the cubic equation

$$ax^3+3bx^2+3cx+d=0.$$

Hence solve the equation

$$x^3-15x^2+33x+847=0.$$

§ 1 P. 112, Ex. P. 113 bottom line.

3. (a) Prove that the sums of the similar powers of the roots of an equation can be expressed rationally in terms of the coefficients.

(b) If α, β, γ be the roots of the cubic $x^3+qx+p=0$, find the value of

$$\Sigma\frac{\alpha^2+\beta\gamma}{\beta+\gamma}.$$

§ 2 P. 99, Ex. 3 (v) P. 86.

4. Explain Horner's method of solving the numerical equation. The cubic $x^3-3x+1=0$ has a root between 1 and 2. Calculate it to five places of decimals.

§ 5 P. 209, Q. 2 P. 227.

1954

1. (a) State Descarte's rule of signs.

Show that the equation $x^4-ax^2-bx-c=0$ has one positive root, one negative root and two complex roots, for all positive values of a, b and c.

(b) The cubic

$$a_0x^3+3a_1x^2+3a_2x+a_3=0$$

has three roots α, β, γ. Find the equation whose roots are $(\alpha-\beta)^2$, $(\beta-\gamma)^2$ and $(\gamma-\alpha)^2$, and explain how its coefficients may be used to determine the nature of the roots of the original cubic.

§ 1 P. 150, as in Ex. 1 P. 155, § 13 P. 78.

2. Explain fully Descarte's method of solving the biquadratic

$$ax^4+4bx^3+6cx^2+4dx+e=0.$$

Hence solve the equation

$$x^4-3x^2-42x-40=0.$$

§ 7 P. 140, Ex. 1 P. 141.

3. (a) Prove that the sums of the similar powers of the roots of an equation can be expressed rationally in terms of the coefficients.

(b) State and explain Sturm's theorem.

Find the number and position of the real roots of the equation $x^5-10x^3+6x+1=0$.

§ 2 P. 97, § 3 P. 179, Ex. 6 P. 191.

4. Explain Horner's method of solving the numerical equation. The equation $x^4+12x+7=0$ has a root between -1 and 0. Calculate it correctly to seven decimal places.

§ 5 P. 209, Ex. 1 P. 227.

1955

1. (a) Find the equation whose roots are the roots of

$$x^5+4x^2-x^2+11=0$$

each diminished by 3.

(b) The roots of the cubic

$$x^3+2x^2+3x+1=0$$

are α. β, γ ; from the equation whose roots are

$$\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\alpha^3},\ \frac{1}{\gamma^3}+\frac{1}{\alpha^3}-\frac{1}{\beta^3},\ \frac{1}{\alpha^3}+\frac{1}{\beta^3}-\frac{1}{\gamma^3}.$$

Ex 1 P. 43, Q. 3 (c) P. 69.

2. Explain fully Descarte's method of solving the biquadratic

$$ax^4+4bx^3+6cx^2+4dx+e=0.$$

Hence solve the equation

$$x^4-3x^2-6x-2=0.$$

§ 7 P. 140, as in Ex. 1 P. 141.

3. (a) In any equation

$$x^n+P_1x^{n-1}+P_2x^{n-2}+\ldots+P_{n-1}x+P_n=0,$$

if the first negative term be $-P_r x^{n-r}$ and if the greatest negative coefficient be $-P_k$, then prove that

$$\sqrt[r]{(P_k)}+1$$

is a supperior limit of the positive roots. Hence find a superior limit of the positive roots of the equation

$$x^4-5x^3+40x^2-8x+23=0.$$

(b) Find the nature of the roots of the equation

$$x^4-5x^3+9x^2-7x+2=0.$$

4. Explain Horner's method of solving the numerical equation.

Find to seven decimal places the root of the equation

$$x^4-12x+7=0$$

which lies between 2 and 3.

See P. 209, Ex. 2 P. 220.

1956

1. (a) Prove that imaginary roots enter equations in pairs.

Prove that if two numbers a and b, substituted for x in the polynomial $f(x)$, give results with contrary signs, an odd number of real roots of the equation $f(x)=0$ lies between them ; and if they give results with the same sign, either no real roots or an even number of real roots lies between them.

(b) If α, β, γ be the roots of the cubic $x^3+qx+r=0$, find the value of

$$\Sigma\frac{\alpha^2+\beta\gamma}{\beta+\gamma}.$$

§ 2·2 P. 4, § 2 P. 151, Ex. 3 (v) P. 86.

2. (a) Find the equation of the squared differences of the cubic $a_0x^3+3a_1x^2+3a_2x+a_3=0$, and discuss the criterion of the nature of the roots of the cubic.

(b) If α, β, γ be the roots of the cubic

$$ax^3+3bx^2+3cx+d=0,$$

find the equation whose roots are

$$\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha},\ \frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta},\ \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}.$$

§ 13 P. 79, Ex. 6 P. 72.

3. (a) Explain Cardan's method of solving the cubic equation $ax^3+3bx^2+3cx+d=0$.

Hence solve the equation

$$x^3-18x-35=0.$$

(b) Prove that if in any equation each negative coefficient be taken positively and divided by the sum of all the positive coefficients which precede it, the greatest quotient thus formed increased by unity is a superior limit of the positive roots. Find a superior limit of the positive roots of

$$x^7+4x^6-3x^5+5x^4-9x^3-11x^2+6x-8=0.$$

§ 1 P. 112, Proceed as in Ex. P. 113.

t quadratic in $(t-27)(t-8)=0$.

$\therefore$ $u=3$, $v=2$ etc. **Ans.** $5,\ \dfrac{-5\pm i\sqrt{3}}{2}$.

4. State Sturm's theorem and apply it to find the number and position of the real roots of the equation

$$x^3+x^2+x-100=0.$$

Find by Horner's method the positive roots of the above equation correct to four decimal places.

§ 3 P. 179, Ex. 1 P. 214.